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Binary quadratic forms and sums of triangular numbers

by

ZHI-HONG SUN (Huaian)

1. Introduction. Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. For $a, b, n \in \mathbb{N}$ let

$$t_n(a,b) = |\{\langle x,y\rangle : n = ax(x-1)/2 + by(y-1)/2,\ x,y \in \mathbb{N}\}|.$$

For convenience we also define $t_0(a,b) = 1$ and $t_{-n}(a,b) = 0$ for $n \in \mathbb{N}$. Let

$$\psi(q) = \sum_{k=1}^{\infty} q^{k(k-1)/2} \quad (|q| < 1).$$

Then clearly

$$\psi(q^a)\psi(q^b) = 1 + \sum_{n=1}^{\infty} t_n(a,b)q^n \quad (|q| < 1). \tag{1.1}$$

Ramanujan conjectured and Berndt proved ([1, pp. 302–303]) that

$$q\psi(q)\psi(q^7) = \sum_{n=1}^{\infty} \left(\frac{-28}{n}\right)\frac{q^n}{1-q^n} \quad (|q| < 1), \tag{1.2}$$

where $(\frac{k}{m})$ is the Legendre–Jacobi–Kronecker symbol. According to Berndt ([1]), (1.2) is of extreme interest, and it would appear to be very difficult to prove it without the addition theorem for elliptic integrals. By (1.1), the equality (1.2) is equivalent to

$$t_n(1,7) = \sum_{k\mid n+1,\ 2\nmid k} \left(\frac{k}{7}\right). \tag{1.3}$$

In [5, 7, 8] K. S. Williams and the author proved (1.3) and so (1.2) by using the theory of binary quadratic forms.

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Let $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{\langle x, y\rangle : x, y \in \mathbb{Z}\}$. For $n \in \mathbb{N}$ and $a, b, c \in \mathbb{Z}$ with $a, c > 0$ and $b^2 - 4ac < 0$ let

$$R([a,b,c],n) = |\{\langle x,y\rangle \in \mathbb{Z}^2 : n = ax^2 + bxy + cy^2\}|. \tag{1.4}$$

In [5] the author proved the following result.

THEOREM 1.1 ([5, Theorem 2.1 and Remark 2.1]). *Let $a, b, n \in \mathbb{N}$. Then*

$$4t_n(a,b) = \begin{cases} R\Big(\Big[a, a, \frac{a+b}{4}\Big], 2n + \frac{a+b}{4}\Big) - R\Big([a,0,b], 2n + \frac{a+b}{4}\Big) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 4 \mid a+b, \\ R\Big(\Big[2a, 2a, \frac{a+b}{2}\Big], 4n + \frac{a+b}{2}\Big) \quad \text{if } 4 \mid a+b-2, \\ R([4a, 4a, a+b], 8n+a+b) \quad \text{if } 2 \nmid a+b. \end{cases}$$

Moreover, if $2 \nmid ab$ and $8 \mid a+b$, then

$$R([a,0,b], 2n+(a+b)/4) = \begin{cases} 0 & \text{if } 2 \nmid n + (a+b)/8, \\ R([a,a,(a+b)/4], (8n+a+b)/16) & \text{if } 2 \mid n+(a+b)/8. \end{cases}$$

For 20 values of $\langle a, b\rangle$, the explicit formulas for $t_n(a,b)$ are known. See Table 1.1.

Table 1.1

$t_n(a,b)$	References for formulas for $t_n(a,b)$
$t_n(1,1)$	Legendre [4]
$t_n(1,3), t_n(1,7)$	Ramanujan, Berndt [1, 2], Williams [8]
$t_n(1,15), t_n(3,5)$	Sun, Williams [7]
$t_n(1,2)$	Sun [5, Theorem 3.1]
$t_n(1,4)$	Sun [5, Theorem 3.2]
$t_n(1,5)$	Sun [5, Theorem 3.3]
$t_n(1,9)$	Sun [5, Theorem 3.4]
$t_n(1,13)$	Sun [5, Theorem 3.5]
$t_n(1,25)$	Sun [5, Theorem 3.6]
$t_n(1,37)$	Sun [5, Theorem 3.7]
$t_n(1,11)$	Sun [5, Theorems 4.1 and 4.2]
$t_n(1,19), t_n(1,43)$	Sun [5, Theorem 4.1]
$t_n(1,67), t_n(1,163)$	Sun [5, Theorem 4.1]
$t_n(1,27)$	Sun [5, Theorem 4.3]
$t_n(1,23)$	Sun [5, Theorems 4.4 and 4.5]
$t_n(1,31)$	Sun [5, Theorem 4.4]

In this paper, by developing the theory of binary quadratic forms we completely determine $t_n(a,b)$ for 123 more values of $\langle a,b\rangle$. See Table 1.2. We also show that $t_{2n-2}(7,9)-t_{2n-8}(1,63)$, $t_{2n-2}(5,11)-t_{2n-7}(1,55)$ and $t_{2n-2}(3,13)-t_{2n-5}(1,39)$ are multiplicative functions of $n\in\mathbb{N}$.

Table 1.2

$t_n(a,b)$	Values of $t_n(a,b)$
$t_n(1,b), t_n(2,b/2)$ $(b=6,10,22,58)$	Theorem 2.2
$t_n(1,12), t_n(3,4), t_n(1,28), t_n(4,7)$	Theorem 2.2
$t_n(1,18), t_n(2,9)$	Theorem 2.4
$t_n(1,60), t_n(3,20), t_n(4,15), t_n(5,12)$	Theorem 2.5
$t_n(1,45), t_n(5,9)$	Theorem 2.7
$t_n(1,3b), t_n(3,b)$ $(b=7,11,19,31,59)$	Theorem 5.1
$t_n(a,6m/a)$ $(a\mid 6,\ m=5,7,13,17)$	Theorem 5.2
$t_n(a,10m/a)$ $(a\mid 10,\ m=7,13,19)$	Theorem 5.3
$t_n(1,85), t_n(5,17)$	Theorem 5.4
$t_n(1,133), t_n(7,19)$	Theorem 5.5
$t_n(1,253), t_n(11,23)$	Theorem 5.6
$t_n(a,15m/a)$ $(a\mid 15,\ m=7,11,23)$	Theorem 7.1
$t_n(1,273), t_n(3,91), t_n(7,39), t_n(13,21)$	Theorem 7.2
$t_n(1,357), t_n(3,119), t_n(7,51), t_n(17,21)$	Theorem 7.3
$t_n(1,385), t_n(5,77), t_n(7,55), t_n(11,35)$	Theorem 7.4
$t_n(a,210/a)$ $(a\mid 30)$	Theorem 7.5
$t_n(a,330/a)$ $(a\mid 30)$	Theorem 7.6
$t_n(a,462/a)$ $(a\mid 42)$	Theorem 7.7
$t_n(a,1365/a)$ $(a\mid 105)$	Theorem 8.2
$t_n(1,8)$	Theorem 10.1
$t_n(1,63), t_n(7,9)$	Theorem 10.2

A nonsquare integer d with $d\equiv 0,1 \pmod 4$ is called a *discriminant*. Let d be a discriminant. The *conductor* of d is the largest positive integer $f=f(d)$ such that $d/f^2\equiv 0,1 \pmod 4$. As usual we set $w(d)=1,2,4,6$ according as $d>0, d<-4, d=-4$ or $d=-3$. For $a,b,c\in\mathbb{Z}$ we denote the equivalence class containing the form $ax^2+bxy+cy^2$ by $[a,b,c]$. It is known ([3]) that

$$(1.5)\qquad [a,b,c]=[c,-b,a]=[a,2ak+b,ak^2+bk+c]\quad \text{for } k\in\mathbb{Z}.$$

Let $H(d)$ be the form class group consisting of classes of primitive, integral binary quadratic forms of discriminant d, and let $h(d)=|H(d)|$. For $n\in\mathbb{N}$ and $[a,b,c]\in H(d)$ we define $R([a,b,c],n)$ and $N(n,d)=\sum_{K\in H(d)}R(K,n)$ as in [6]. In particular, when $a>0$ and $b^2-4ac<0$, then $R([a,b,c],n)$ is given by (1.4). It is known that $R([a,b,c],n) = R([a,-b,c],n)$. If $R([a,b,c],n)>0$, we say that $n\in R([a,b,c])$, n is represented by $[a,b,c]$ or $ax^2+bxy+cy^2$, and write $n=ax^2+bxy+cy^2$.

Let $n\in\mathbb{N}$ and let d be a negative discriminant such that $h(d)=4$. For $m\in\mathbb{N}$ let C_m be the cyclic group of order m. Then $H(d)\cong C_4$ or $H(d)\cong C_2\times C_2$. If $H(d)\cong C_4$ with generator A, for 50 such discriminants d, in Section 9 we give explicit formulas for $R(A,n)$. When $H(d)\cong C_2\times\cdots\times C_2$, in Sections 2, 3, 4, 6 and 8 we determine $R(K,n)$ for any $K\in H(d)$. As applications, in Sections 2, 5, 7, 8 and 10 we deduce many explicit formulas for $t_n(a,b)$.

In addition to the above notation, throughout this paper $[x]$ denotes the greatest integer not exceeding x, $\mu(n)$ denotes the Möbius function, (a,b) denotes the greatest common divisor of integers a and b. For a prime p and $n\in\mathbb{N}$, $\mathrm{ord}_p\, n$ denotes the unique nonnegative integer α such that $p^\alpha \parallel n$ (i.e. $p^\alpha\mid n$ but $p^{\alpha+1}\nmid n$).

Throughout this paper p denotes a prime and products (sums) over p run through all distinct primes p satisfying any restrictions given under the product (summation) symbol. For example the condition $p\equiv 1 \pmod 4$ under a product restricts the product to those distinct primes p which are of the form $4k+1$.

2. Formulas for $t_n(1,b)$ in the cases $b=6,10,12,18,22,28,45,58,60$. Let d be a discriminant and $n\in\mathbb{N}$. In view of [6, Lemma 4.1], we introduce

$$(2.1)\qquad \delta(n,d)=\sum_{k\mid n}\Big(\frac{d}{k}\Big) = \begin{cases} \prod_{(\frac{d}{p})=1}(1+\mathrm{ord}_p\, n) & \text{if } 2\mid \mathrm{ord}_q\, n \text{ for every prime } q \text{ with } (\frac{d}{q})=-1,\\ 0 & \text{otherwise.}\end{cases}$$

We recall that $N(n,d)=\sum_{K\in H(d)}R(K,n)$.

Lemma 2.1 ([6, Theorem 4.1]). *Let d be a discriminant with conductor f. Let $n\in\mathbb{N}$ and $d_0=d/f^2$. Then*

$$N(n,d)=\begin{cases} 0 & \text{if } (n,f^2) \text{ is not a square,}\\ m\prod_{p\mid m}\Big(1-\frac{1}{p}\Big(\frac{d/m^2}{p}\Big)\Big)\cdot w(d)\delta\Big(\frac{n}{m^2},d_0\Big) & \text{if } (n,f^2)=m^2 \text{ for } m\in\mathbb{N}.\end{cases}$$

In particular, when $(n,f)=1$ we have $N(n,d)=w(d)\delta(n,d_0)$.

As $f(d_0)=1$, by Lemma 2.1 we have $N(n/m^2,d_0)=w(d_0)\delta(n/m^2,\,d_0)$. Thus, if $(n,f^2)=m^2$ for $m\in\mathbb{N}$, using Lemma 2.1 we see that

$$(2.2)\qquad \frac{N(n,d)}{w(d)} = m\prod_{p\mid m}\left(1-\frac{1}{p}\left(\frac{d/m^2}{p}\right)\right)\cdot\frac{N(n/m^2,d_0)}{w(d_0)}.$$

This is a reduction formula for $N(n,d)$.

LEMMA 2.2. *Let $a,b,n\in\mathbb{N}$ with $2\nmid n$.*

(i) *If $2\nmid a$ and $4\nmid(a-b)b$, then*

$$R([a,0,4b],n) = \begin{cases} R([a,0,b],n) & \text{if } n\equiv a \pmod 4,\\ 0 & \text{otherwise.}\end{cases}$$

If $2\nmid a$, $2\mid b$ and $8\nmid b$, then

$$R([a,0,4b],n) = \begin{cases} R([a,0,b],n) & \text{if } n\equiv a \pmod 8,\\ 0 & \text{otherwise.}\end{cases}$$

(ii) *If $2\nmid a+b$ and $8\nmid ab$, then*

$$R([4a,4a,a+b],n) = \begin{cases} R([a,0,b],n) & \text{if } n\equiv a+b \pmod 8,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Suppose $2\nmid a$ and $4\nmid(a-b)b$. Clearly $n=ax^2+4by^2$ implies $2\nmid x$ and so $n\equiv a\pmod 4$. Now assume $n\equiv a\pmod 4$ and $n=ax^2+by^2$. If $2\nmid y$, then $a-b\equiv n-by^2=ax^2\pmod 4$ and so $4\mid(a-b)b$. This contradicts the assumption. Thus $n=ax^2+by^2$ implies $2\mid y$. Therefore $R([a,0,b],n)=R([a,0,4b],n)$.

Suppose $2\nmid a$, $2\mid b$ and $8\nmid b$. As $(2m+1)^2\equiv 1\pmod 8$ we see that

$$\begin{aligned} R([a,0,4b],n) &= |\{\langle x,y\rangle\in\mathbb{Z}^2 : n=ax^2+by^2,\ 2\mid y\}|\\ &= \begin{cases} R([a,0,b],n) & \text{if } n\equiv a\pmod 8,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This proves (i).

Now let us consider (ii). Assume that $2\nmid a+b$ and $8\nmid ab$. Then

$$\begin{aligned} R([4a,4a,a+b],n) &= |\{\langle x,y\rangle\in\mathbb{Z}^2 : n=a(2x+y)^2+by^2\}|\\ &= |\{\langle x,y\rangle\in\mathbb{Z}^2 : n=ax^2+by^2,\ 2\mid x-y\}|\\ &= |\{\langle x,y\rangle\in\mathbb{Z}^2 : n=ax^2+by^2,\ 2\nmid xy\}|.\end{aligned}$$

As $(2m+1)^2\equiv 1\pmod 8$, from the above we see that $R([4a,4a,a+b],n)=0$ provided $n\not\equiv a+b\pmod 8$. Now assume $n\equiv a+b\pmod 8$. If $n=ax^2+by^2$ with $x,y\in\mathbb{Z}$ and $2\mid xy$, then $n\equiv a,a+4b,b$ or $b+4a\pmod 8$. Since $8\nmid ab$ we have $a+b\not\equiv a,a+4b,b,b+4a\pmod 8$ and so $n\not\equiv a+b\pmod 8$. Thus, if $n=ax^2+by^2$ for some $x,y\in\mathbb{Z}$, we must have $2\nmid xy$. Hence, from the above we deduce $R([4a,4a,a+b],n)=R([a,0,b],n)$. So (ii) is true and the lemma is proved.

THEOREM 2.1. *Let* $b \in \{6,10,12,22,28,58\}$, $b = 2^r b_0$ $(2 \nmid b_0)$, $n \in \mathbb{N}$ *and* $2 \nmid n$. *Then*

$$R([1,0,4b],n) = \begin{cases} 2\sum_{k\mid n}\left(\frac{-b}{k}\right) & \text{if } n \equiv 1 \pmod 8, \\ 0 & \text{otherwise}, \end{cases}$$

$$R([4,4,b+1],n) = \begin{cases} 2\sum_{k\mid n}\left(\frac{-b}{k}\right) & \text{if } n \equiv b+1 \pmod 8, \\ 0 & \text{otherwise}, \end{cases}$$

$$R([2^{r+2},0,b_0],n) = \begin{cases} 2\sum_{k\mid n}\left(\frac{-b}{k}\right) & \text{if } n \equiv b_0 \pmod 8, \\ 0 & \text{otherwise}, \end{cases}$$

$$R([2^{r+2},2^{r+2},2^r+b_0],n) = \begin{cases} 2\sum_{k\mid n}\left(\frac{-b}{k}\right) & \text{if } n \equiv 2^r+b_0 \pmod 8, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. As $2 \mid b$, $2 \nmid b_0$, $r \in \{1,2\}$ and $2 \nmid n$, using Lemma 2.2 we see that

$$R([1,0,4b],n) = \begin{cases} R([1,0,b],n) & \text{if } 8 \mid n-1, \\ 0 & \text{otherwise}, \end{cases}$$

$$R([2^{r+2},0,b_0],n) = \begin{cases} R([2^r,0,b_0],n) & \text{if } 8 \mid n-b_0, \\ 0 & \text{otherwise}, \end{cases}$$

$$R([4,4,b+1],n) = \begin{cases} R([1,0,b],n) & \text{if } n \equiv b+1 \pmod 8, \\ 0 & \text{otherwise}, \end{cases}$$

$$R([2^{r+2},2^{r+2},2^r+b_0],n) = \begin{cases} R([2^r,0,b_0],n) & \text{if } n \equiv 2^r+b_0 \pmod 8, \\ 0 & \text{otherwise}. \end{cases}$$

It is known that $H(-4b) = \{[1,0,b],[2^r,0,b_0]\}$. See [6, Table 9.1]. Clearly $n = x^2+by^2$ implies $n \equiv 1, b+1 \pmod 8$, and $n = 2^r x^2 + b_0 y^2$ implies $n \equiv b_0, 2^r+b_0 \pmod 8$. Since $1, b+1, b_0, 2^r+b_0$ are distinct modulo 8, we see that

$$N(n,-4b) = \begin{cases} R([1,0,b],n) & \text{if } n \equiv 1, b+1 \pmod 8, \\ R([2^r,0,b_0],n) & \text{if } n \equiv b_0, 2^r+b_0 \pmod 8. \end{cases}$$

Since $f(-4b) \in \{1,4\}$ and $2 \nmid n$, we have $(n, f(-4b)) = 1$. Hence, by Lemma 2.1 we have

$$N(n,-4b) = w(-4b)\sum_{k\mid n}\left(\frac{-4b}{k}\right) = 2\sum_{k\mid n}\left(\frac{-b}{k}\right).$$

Now putting all the above together we deduce the result.

For $b \in \{6,10,12,22,28,58\}$ set $b = 2^r b_0$ $(2 \nmid b_0)$. From Theorem 1.1 we easily see that $4t_n(1,b) = R([4,4,b+1],8n+b+1)$ and $4t_n(2^r,b_0) =$

$R([2^{r+2},2^{r+2},2^r+b_0],8n+2^r+b_0)$. Thus, applying Theorem 2.1 we deduce the following result.

THEOREM 2.2. *Let $n\in\mathbb{N}$ and $b\in\{6,10,12,22,28,58\}$.*

(i) *If $b\in\{6,10,22,58\}$, then*

$$t_n(1,b)=\frac{1}{2}\sum_{k\mid 8n+b+1}\Big(\frac{-b}{k}\Big)\quad\text{and}\quad t_n(2,b/2)=\frac{1}{2}\sum_{k\mid 8n+2+b/2}\Big(\frac{-b}{k}\Big).$$

(ii) *If $b\in\{12,28\}$, then*

$$t_n(1,b)=\frac{1}{2}\sum_{k\mid 8n+b+1}\Big(\frac{k}{b/4}\Big)\quad\text{and}\quad t_n(4,b/4)=\frac{1}{2}\sum_{k\mid 8n+4+b/4}\Big(\frac{k}{b/4}\Big).$$

THEOREM 2.3. *Let $a,b,n\in\mathbb{N}$.*

(i) *If $8\nmid a$, $8\nmid b$ and $4\nmid a+b$, then $t_n(a,b)=\frac{1}{4}R([a,0,b],8n+a+b)$.*
(ii) *If $2\nmid a, 8\mid b-4$ and $4\mid a+b/4$, then $t_n(a,b)=\frac{1}{4}R([a,0,b/4],8n+a+b)$.*

Proof. As $x(x-1)/2=(1-x)(1-x-1)/2$, we see that

$$\begin{aligned}4t_n(a,b)&=|\{\langle x,y\rangle\in\mathbb{Z}^2: n=a(x^2-x)/2+b(y^2-y)/2\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2: 8n+a+b=a(2x-1)^2+b(2y-1)^2\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2: 8n+a+b=ax^2+by^2,\ 2\nmid xy\}|.\end{aligned}$$

Let $x,y\in\mathbb{Z}$ be such that $8n+a+b=ax^2+by^2$. When $2\mid x$ and $2\mid y$, we have $4\mid 8n+a+b$ and so $4\mid a+b$. When $2\mid x$ and $2\nmid y$, we have $a\equiv 8n+a=ax^2+by^2-b\equiv ax^2\equiv 0,4a\pmod 8$ and so $8\mid a$. When $2\nmid x$ and $2\mid y$, we have $b\equiv 8n+b=ax^2+by^2-a\equiv by^2\equiv 0,4b\pmod 8$ and so $8\mid b$. Thus, if $8\nmid a$, $8\nmid b$ and $4\nmid a+b$, by the above we must have $2\nmid xy$. Hence,

$$\begin{aligned}4t_n(a,b)&=|\{\langle x,y\rangle\in\mathbb{Z}^2: 8n+a+b=ax^2+by^2,\ 2\nmid xy\}|\\&=R([a,0,b],8n+a+b).\end{aligned}$$

This proves (i).

Now assume $2\nmid a$, $4\mid b$, and $4\mid a+b/4$. Set $b=4b_0$. Then $2\nmid b_0$ and $4\nmid a-b_0$. As $8n+a+4b_0=ax^2+4b_0y^2$ $(x,y\in\mathbb{Z})$ implies $2\nmid xy$, and $8n+a+4b_0=ax^2+b_0y^2$ $(x,y\in\mathbb{Z})$ implies $2\mid y$, from the above we deduce

$$\begin{aligned}4t_n(a,b)&=|\{\langle x,y\rangle\in\mathbb{Z}^2: 8n+a+4b_0=ax^2+4b_0y^2,\ 2\nmid xy\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2: 8n+a+4b_0=ax^2+4b_0y^2\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2: 8n+a+4b_0=ax^2+b_0y^2\}|.\end{aligned}$$

This completes the proof.

THEOREM 2.4. *Let $n \in \mathbb{N}$ and $a \in \{1,2\}$. Then*

$$t_n(a,18/a) = \begin{cases} \frac{1}{2}\sum_{k\mid 8n+a+18/a}\left(\frac{-2}{k}\right) & \text{if } 3\mid n, \\ \frac{1}{2}\sum_{k\mid (8n+a+18/a)/9}\left(\frac{-2}{k}\right) & \text{if } 9\mid n-a, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Theorem 2.3 yields $4t_n(a,18/a) = R([a,0,18/a], 8n+a+18/a)$. As $H(-72) = \{[1,0,18],[2,0,9]\}$ and $f(-72) = 3$, from [6, Theorem 9.3] and (2.1) we see that

$$R([a,0,18/a],8n+a+18/a)$$
$$= \begin{cases} \left(1-(-1)^a\left(\frac{8n+a}{3}\right)\right)\sum_{k\mid 8n+a+18/a}\left(\frac{-8}{k}\right) & \text{if } 3\nmid 8n+a, \\ 2\sum_{k\mid (8n+a+18/a)/9}\left(\frac{-8}{k}\right) & \text{if } 9\mid 8n+a, \\ 0 & \text{otherwise.} \end{cases}$$

So the result follows.

REMARK 2.1. Theorem 2.2 can also be proved by using Theorem 2.3 instead of Theorem 1.1. When $b \in \{6,10,12,18,22,28,58\}$ and $8n+b+1$ is a prime power, the formulas for $t_n(1,b)$ have been given by the author in [5, Theorems 5.1 and 5.3]. When $b \in \{3,5,9,11,29\}$ and $8n+b+2$ is a prime power, the formulas for $t_n(2,b)$ have been given by the author in [5, Theorems 5.2 and 5.4].

For $b \in \{5,13\}$ set $4n+(b+1)/2 = b^\alpha n_0$ ($b\nmid n_0$). Clearly the fact that $2\mid \mathrm{ord}_p(4n+(b+1)/2)$ for every odd prime p with $(\frac{-b}{p}) = -1$ implies $(\frac{-b}{n_0}) = \prod_{p\mid n_0}(\frac{-b}{p})^{\mathrm{ord}_p n_0} = 1$ and so $(\frac{n_0}{b}) = (\frac{b}{n_0}) = (\frac{-1}{n_0}) = (-1)^{(b+1)/2} = -1$. From this and (2.1) we see that Theorems 3.3 and 3.5 in [5] can be rewritten as

$$t_n(1,5) = \frac{1}{2}\sum_{k\mid 4n+3}\left(\frac{-5}{k}\right) \quad\text{and}\quad t_n(1,13) = \frac{1}{2}\sum_{k\mid 4n+7}\left(\frac{-13}{k}\right). \tag{2.3}$$

We also note that for $a,b,n \in \mathbb{N}$,

$$\begin{aligned}&|\{\langle x,y\rangle \in \mathbb{Z}^2 : n = a(2x^2-x)+b(2y^2-y)\}| \\ &\quad= |\{\langle x,y\rangle \in \mathbb{Z}^2 : 8n+a+b = a(4x-1)^2+b(4y-1)^2\}| \\ &\quad= \frac{1}{4}|\{\langle x,y\rangle \in \mathbb{Z}^2 : 8n+a+b = ax^2+by^2,\ 2\nmid xy\}| = t_n(a,b).\end{aligned}$$

THEOREM 2.5. *Let $n \in \mathbb{N}$, $a \in \{1,3,5,15\}$ and $8n+a+60/a = 3^\alpha n_0$ ($3\nmid n_0$). Then*

$$t_n(a,60/a)=\begin{cases}\frac14\Big(1+(-1)^\alpha\Big(\frac{n_0}{3}\Big)\Big)\sum_{k\mid n_0}\Big(\frac{k}{15}\Big) & \text{if } a=1,15,\\ \frac14\Big(1-(-1)^\alpha\Big(\frac{n_0}{3}\Big)\Big)\sum_{k\mid n_0}\Big(\frac{k}{15}\Big) & \text{if } a=3,5.\end{cases}$$

Proof. From Theorem 2.3(ii) we see that $4t_n(a,60/a)=R([a,0,15/a],8n+a+60/a)$. By [6, Theorem 9.3] and (2.1) we have

$$R([a,0,15/a],8n+a+60/a)=\begin{cases}\Big(1+(-1)^\alpha\Big(\frac{n_0}{3}\Big)\Big)\sum_{k\mid 3^\alpha n_0}\Big(\frac{-15}{k}\Big) & \text{if } a=1,15,\\ \Big(1-(-1)^\alpha\Big(\frac{n_0}{3}\Big)\Big)\sum_{k\mid 3^\alpha n_0}\Big(\frac{-15}{k}\Big) & \text{if } a=3,5.\end{cases}$$

Now combining all the above with the fact that $\sum_{k\mid 3^\alpha n_0}(\frac{-15}{k})=\sum_{k\mid n_0}(\frac{k}{15})$ we deduce the result.

THEOREM 2.6. *Let $n\in\mathbb{N}$ and $n=5^\alpha n_0$ $(5\nmid n_0)$. Then*

$$R([1,0,45],n)=\begin{cases}2\sum_{k\mid n_0/9}\Big(\frac{-20}{k}\Big) & \text{if } 9\mid n \text{ and } n_0\equiv\pm1 \pmod 5,\\ 2\sum_{k\mid n_0}\Big(\frac{-20}{k}\Big) & \text{if } 3\mid n-1 \text{ and } n_0\equiv\pm1 \pmod 5,\\ 0 & \text{otherwise},\end{cases}$$

$$R([5,0,9],n)=\begin{cases}2\sum_{k\mid n_0/9}\Big(\frac{-20}{k}\Big) & \text{if } 9\mid n \text{ and } n_0\equiv\pm1 \pmod 5,\\ 2\sum_{k\mid n_0}\Big(\frac{-20}{k}\Big) & \text{if } 3\mid n-2 \text{ and } n_0\equiv\pm1 \pmod 5,\\ 0 & \text{otherwise}.\end{cases}$$

Proof. It is known that $f(-180)=3$ and $H(-180)=\{[1,0,45],[5,0,9],[2,2,23],[7,4,7]\}$. If $n\equiv2\pmod3$ or $n\equiv\pm3\pmod9$, then clearly $n=x^2+45y^2$ is insolvable and so $R([1,0,45],n)=0$. If $9\mid n$, then clearly

$$\begin{aligned}R([1,0,45],n)&=R([1,0,5],n/9)=R([1,0,5],5^\alpha n_0/9)\\&=R([1,0,5],5^{\alpha-1}n_0/9)=\cdots=R([1,0,5],n_0/9).\end{aligned}$$

It is known that $f(-20)=1$ and $H(-20)=\{[1,0,5],[2,2,3]\}$. Thus, if $n_0\equiv\pm2\pmod5$, then $n_0/9\equiv\mp2\pmod5$ and so $R([1,0,45],n)=R([1,0,45],n_0/9)=0$; if $n_0\equiv\pm1\pmod5$, as $n_0/9=2x^2+2xy+3y^2$ implies $2n_0/9=(2x+y)^2+5y^2\equiv\pm1\pmod5$ and so $n_0\equiv\pm2\pmod5$, we have $R([2,2,3],n_0/9)=0$ and so

$$R([1,0,45],n) = R([1,0,5],n_0/9) = N(n_0/9,-20) = 2\sum_{k\mid n_0/9}\Big(\frac{-20}{k}\Big).$$

Now we assume $n \equiv 1 \pmod 3$. Then $3 \nmid n_0$. For $m \in \mathbb{N}$, $5m = x^2 + 45y^2$ implies $5 \mid x$, and $5m = 5x^2 + 9y^2$ implies $5 \mid y$. Thus $R([1,0,45],5m) = R([5,0,9],m)$ and $R([5,0,9],5m) = R([1,0,45],m)$. Therefore,

$$\begin{aligned} R([1,0,45],n) &= R([1,0,45],5^\alpha n_0) = R([5,0,9],5^{\alpha-1}n_0) \\ &= R([1,0,45],5^{\alpha-2}n_0) = \cdots = \begin{cases} R([1,0,45],n_0) & \text{if } 2 \mid \alpha, \\ R([5,0,9],n_0) & \text{if } 2 \nmid \alpha. \end{cases} \end{aligned}$$

If $n_0 \equiv \pm 2 \pmod 5$, then n_0 cannot be represented by x^2+45y^2 and $5x^2+9y^2$. Thus $R([1,0,45],n) = 0$ by the above. Now suppose $n_0 \equiv \pm 1 \pmod 5$. It is easily seen that n_0 cannot be represented by $[2,2,23]$ and $[7,4,7]$. Clearly $n_0 = x^2 + 45y^2$ implies $n_0 \equiv 1 \pmod 3$, and $n_0 = 5x^2 + 9y^2$ implies $n_0 \equiv 2 \pmod 3$. Since $n_0 = 5^{-\alpha} n \equiv (-1)^\alpha n \equiv (-1)^\alpha \pmod 3$, using the above and Lemma 2.1 we see that

$$\begin{aligned} R([1,0,45],n) &= \begin{cases} R([1,0,45],n_0) & \text{if } n_0 \equiv 1 \pmod 3, \\ R([5,0,9],n_0) & \text{if } n_0 \equiv 2 \pmod 3 \end{cases} \\ &= N(n_0,-180) = 2\sum_{a\mid n_0}\Big(\frac{-20}{a}\Big). \end{aligned}$$

Combining all the above we prove the formula for $R([1,0,45],n)$. Since $R([1,0,45],5n) = R([5,0,9],n)$, replacing n with $5n$ in the formula for $R([1,0,45],n)$ we deduce the result for $R([5,0,9],n)$. This completes the proof.

Theorem 2.7. *Let $n \in \mathbb{N}$.*

(i) *If $4n + 23 = 5^\alpha n_1$ $(5 \nmid n_1)$, then*

$$t_n(1,45) = \begin{cases} \frac{1}{2}\sum_{k\mid n_1/9}(-1)^{(k-1)/2}\Big(\frac{k}{5}\Big) & \text{if } 9 \mid n-1 \text{ and } n_1 \equiv \pm 2 \pmod 5, \\ \frac{1}{2}\sum_{k\mid n_1}(-1)^{(k-1)/2}\Big(\frac{k}{5}\Big) & \text{if } 3 \mid n \text{ and } n_1 \equiv \pm 2 \pmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $4n + 7 = 5^\alpha n_1$ $(5 \nmid n_1)$, then*

$$t_n(5,9) = \begin{cases} \frac{1}{2}\sum_{k\mid n_1/9}(-1)^{(k-1)/2}\Big(\frac{k}{5}\Big) & \text{if } 9 \mid n-5 \text{ and } n_1 \equiv \pm 2 \pmod 5, \\ \frac{1}{2}\sum_{k\mid n_1}(-1)^{(k-1)/2}\Big(\frac{k}{5}\Big) & \text{if } 3 \mid n \text{ and } n_1 \equiv \pm 2 \pmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Theorem 2.3(i) we have

$$4t_n(1,45) = R([1,0,45],8n+46) \quad \text{and} \quad 4t_n(5,9) = R([5,0,9],8n+14).$$

Thus applying Theorem 2.6 we deduce the result.

From Theorem 2.7, (2.1) and the fact that $(-1)^{(k-1)/2}(\frac{k}{5}) = (\frac{-20}{k})$ we have the following corollaries.

COROLLARY 2.1. *Let $n \in \mathbb{N}$ and $4n+23 = 5^\alpha n_1$ with $5 \nmid n_1$. Then n is represented by $\frac{x(x-1)}{2}+45\frac{y(y-1)}{2}$ if and only if $9 \mid n^2(n-1)$, $n_1 \equiv \pm 2 \pmod 5$ and $2 \mid \mathrm{ord}_q n_1$ for every prime $q \equiv 11,13,17,19 \pmod{20}$.*

COROLLARY 2.2. *Let $n \in \mathbb{N}$ and $4n+7 = 5^\alpha n_1$ with $5 \nmid n_1$. Then n is represented by $5\frac{x(x-1)}{2}+9\frac{y(y-1)}{2}$ if and only if $9 \mid n^2(n-5)$, $n_1 \equiv \pm 2 \pmod 5$ and $2 \mid \mathrm{ord}_q n_1$ for every prime $q \equiv 11,13,17,19 \pmod{20}$.*

3. General results for $R(K,n)$ when $K \in H(d)$ and $H(d) \cong C_2 \times \cdots \times C_2$. Let d be a discriminant with conductor f and $d_0 = d/f^2$. Assume $k,m \in \mathbb{N}$, $k \mid d_0$, $4 \nmid k$, $m \mid f$ and $(k, f/m) = 1$. By [6, Lemma 2.1], for any $K \in H(d)$ there exist $a,b,c \in \mathbb{Z}$ such that $K = [a, bkm, ckm^2]$ with $(a, km) = 1$ and $(c,k) = 1$. Following [6, Definition 2.1] we define $\varphi_{k,m}(K) = [ak, bk, c]$.

By the definition, for any $[a, bm, cm^2] \in H(d)$ and $[a, bk, ck] \in H(d)$ with $(c,k) = 1$ we have

$$\varphi_{1,m}([a,bm,cm^2]) = [a,b,c], \qquad \varphi_{k,1}([a,bk,ck]) = [ak,bk,c]$$

and

$$\varphi_{k,m}(K) = \varphi_{k,1}(\varphi_{1,m}(K)) \quad \text{ for } K \in H(d).$$

From [6, Theorem 2.1] we know that $\varphi_{1,m}$ is a surjective homomorphism from $H(d)$ to $H(d/m^2)$.

Let d be a discriminant and $H^2(d) = \{K^2 : K \in H(d)\}$. Let $G(d) = H(d)/H^2(d)$ denote the group of genera, and let $\omega(d)$ denote the number of distinct prime divisors of d. It is well known ([3]) that $|G(d)| = 2^{t(d)}$, where

$$t(d) = \begin{cases} \omega(d) & \text{if } d \equiv 0 \pmod{32}, \\ \omega(d)-2 & \text{if } d \equiv 4 \pmod{16}, \\ \omega(d)-1 & \text{otherwise.} \end{cases}$$

LEMMA 3.1 ([6, Theorem 6.1]). *Let d be a discriminant with conductor f, $d_0 = d/f^2$ and $n \in \mathbb{N}$. If (n, f^2) is not a square, or there exists a prime p such that $2 \nmid \mathrm{ord}_p n$ and $(\frac{d_0}{p}) = -1$, then $R(G,n) = 0$ for any $G \in G(d)$. Suppose $(n,f^2) = m^2$ for $m \in \mathbb{N}$ and $(\frac{d_0}{p}) = 0,1$ for every prime p with $2 \nmid \mathrm{ord}_p n$. Then there are exactly $2^{t(d)-t(d/m^2)}$ genera G representing n, and for such a genus G we have $R(G,n) = N(n,d)/2^{t(d)-t(d/m^2)}$.*

Lemma 3.2. *Let d be a discriminant with conductor f and $|H^2(d)|=1$. For any positive divisor m of f we have $|H^2(d/m^2)|=1$ and*

$$m\prod_{p\mid m}\Big(1-\frac{1}{p}\Big(\frac{d/m^2}{p}\Big)\Big)=\begin{cases}2^{t(d)-t(d/m^2)}\frac{w(d/m^2)}{w(d)} & \text{if } d<0,\\ 2^{t(d)-t(d/m^2)}\frac{\log\varepsilon(d)}{\log\varepsilon(d/m^2)} & \text{if } d>0,\end{cases}$$

where $\varepsilon(d)=(x_1+y_1\sqrt{d})/2$ and (x_1,y_1) is the solution in positive integers to the equation $x^2-dy^2=4$ for which $x_1+y_1\sqrt{d}$ is least.

Proof. From [6, Theorem 2.1 and Lemma 2.6(i)] we know that $\varphi_{1,m}$ is a surjective homomorphism from $H^2(d)$ to $H^2(d/m^2)$. Since $|H^2(d)|=1$, we must have $|H^2(d/m^2)|=1$. As $|H^2(d)|=1$, we have $G(d)=H(d)/H^2(d)\cong H(d)$ and so $h(d)=|G(d)|=2^{t(d)}$. Since $|H^2(d/m^2)|=1$, we have $G(d/m^2)=H(d/m^2)/H^2(d/m^2)\cong H(d/m^2)$ and so $h(d/m^2)=|G(d/m^2)|=2^{t(d/m^2)}$. Now applying the above and [6, Lemma 3.5] we obtain the result.

Theorem 3.1. *Let d be a discriminant with conductor f, $d_0=d/f^2$ and $|H^2(d)|=1$. Let $K\in H(d)$ and $n\in\mathbb{N}$ with $R(K,n)>0$. Then $(n,f^2)=m^2$ for some $m\in\mathbb{N}$ and*

$$R(K,n)=\begin{cases}w\Big(\frac{d}{m^2}\Big)\prod_{(\frac{d_0}{p})=1}\Big(1+\operatorname{ord}_p\frac{n}{m^2}\Big) & \text{if } d<0,\\ \frac{1}{2^{t(d)-t(d/m^2)}}\cdot m\prod_{p\mid m}\Big(1-\frac{1}{p}\Big(\frac{d/m^2}{p}\Big)\Big) & \\ \qquad\times\prod_{(\frac{d_0}{p})=1}\Big(1+\operatorname{ord}_p\frac{n}{m^2}\Big) & \text{if } d>0.\end{cases}$$

Proof. Since $|H^2(d)|=1$, every genus consists of a single class. Thus, applying Lemmas 2.1 and 3.1 we have $(n,f^2)=m^2$ for some $m\in\mathbb{N}$ and

$$\begin{aligned}R(K,n)&=\frac{1}{2^{t(d)-t(d/m^2)}}N(n,d)\\&=\frac{w(d)}{2^{t(d)-t(d/m^2)}}\cdot m\prod_{p\mid m}\Big(1-\frac{1}{p}\Big(\frac{d/m^2}{p}\Big)\Big)\prod_{(\frac{d_0}{p})=1}\Big(1+\operatorname{ord}_p\frac{n}{m^2}\Big).\end{aligned}$$

This together with Lemma 3.2 gives the result.

Euler called a positive integer n a *convenient number* if it satisfies the following criterion: *Let m be an odd number such that $(m,n)=1$ and $m=x^2+ny^2$ with $(x,y)=1$. If the equation $m=x^2+ny^2$ has only one solution with $x,y\ge 0$, then m is a prime.*

According to [3], Euler listed 65 convenient numbers as in Table 3.1 below. He was interested in convenient numbers because they helped him find large primes. Gauss observed that a positive integer n is a convenient number if and only if $|H^2(-4n)| = 1$. In 1973 it was known that Euler's list is complete except for possibly one more n.

Table 3.1

$h(-4n)$	n's with $\lvert H^2(-4n)\rvert = 1$
1	$1, 2, 3, 4, 7$
2	$5, 6, 8, 9, 10, 12, 13, 15, 16, 18, 22, 25, 28, 37, 58$
4	$21, 24, 30, 33, 40, 42, 45, 48, 57, 60, 70, 72, 78,$ $85, 88, 93, 102, 112, 130, 133, 177, 190, 232, 253$
8	$105, 120, 165, 168, 210, 240, 273, 280, 312, 330,$ $345, 357, 385, 408, 462, 520, 760$
16	$840, 1320, 1365, 1848$

THEOREM 3.2. *Let $d < 0$ be a discriminant with conductor f and $d_0 = d/f^2$. Let $d_1 \in \mathbb{N}$, $d_1 \mid d_0$ and $K \in H(d)$. Let $(n, f^2) = m^2$ for $m \in \mathbb{N}$. Let $k = \prod_{p \mid d_1,\, 2 \nmid \mathrm{ord}_p n} p$ and $n_1 = \prod_{p \nmid d_1} p^{\mathrm{ord}_p(n/m^2)}$. Then $R(K, n) = R(\varphi_{k,m}(K), n_1)$.*

Proof. By [6, Theorem 3.2] we have $R(K, n) = R(K', n/m^2)$, where $K' = \varphi_{1,m}(K) \in H(d/m^2)$. Let p be a prime such that $p \mid d_1$ and $p \mid n/m^2$. Since $(n/m^2, f^2/m^2) = 1$ we have $(n/m^2, f/m) = 1$ and so $p \nmid f/m$. By [6, Lemma 2.1] we may assume $K' = [a, bp, cp]$ with $a, b, c \in \mathbb{Z}$ and $p \nmid ac$. Suppose $\alpha_p = \mathrm{ord}_p(n/m^2)$. Then $2 \nmid \alpha_p$ if and only if $2 \nmid \mathrm{ord}_p n$. Applying [6, Lemma 3.4] we have

$$R\left(K', \frac{n}{m^2}\right) = R\left(K', \frac{n/m^2}{p^2}\right) = \cdots = R\left(K', \frac{n/m^2}{p^{2[\alpha_p/2]}}\right).$$

Thus,

$$R\left(K', \frac{n}{m^2}\right) = \cdots = R\left(K', \frac{n/m^2}{\prod_{p \mid d_1,\, p \mid n/m^2} p^{2[\alpha_p/2]}}\right) = R(K', kn_1).$$

From the above and [6, Lemma 3.4] we deduce

$$\begin{aligned} R(K, n) &= R(K', n/m^2) = R(K', kn_1) = R(\varphi_{k,1}(K'), n_1) \\ &= R(\varphi_{k,1}(\varphi_{1,m}(K)), n_1) = R(\varphi_{k,m}(K), n_1). \end{aligned}$$

This is the result.

LEMMA 3.3 ([6, Theorem 2.2]). *Let d be a discriminant with conductor f and $d_0 = d/f^2$. Suppose $k \in \mathbb{N}$, $k \mid d_0$, $4 \nmid k$ and $(k, f) = 1$. For $K \in H(d)$ we have*

$$\varphi_{k,1}(K) = \begin{cases} [k, 0, -d/(4k)]K & \text{if } 4k \mid d, \\ [k, k, (k^2 - d)/(4k)]K & \text{if } 4k \nmid d. \end{cases}$$

Theorem 3.3. *Let d be a discriminant with conductor f and $d_0 = d/f^2$. Let $k \in \mathbb{N}$ be squarefree, $k \mid d_0$ and $(k, f) = 1$. Let $n \in \mathbb{N}$,*

$$I = \begin{cases} \left[1, 0, \dfrac{-d}{4}\right] & \text{if } 4 \mid d, \\ \left[1, 1, \dfrac{1-d}{4}\right] & \text{if } 4 \nmid d \end{cases} \quad \text{and} \quad K = \begin{cases} \left[k, 0, \dfrac{-d}{4k}\right] & \text{if } 4k \mid d, \\ \left[k, k, \dfrac{k^2-d}{4k}\right] & \text{if } 4k \nmid d. \end{cases}$$

Then $R(K, n) = R(I, kn)$.

Proof. By Lemma 3.3 we have $\varphi_{k,1}(I) = K$. Thus applying [6, Lemma 3.4] we obtain $R(I, kn) = R(\varphi_{k,1}(I), n) = R(K, n)$ as asserted.

4. Formulas for $R(K, n)$ when $K \in H(d)$, $d < 0$, $f(d) = 1$ and $H(d) \cong C_2 \times C_2$. Let $d < 0$ be a discriminant. Then $H(d) \cong C_2 \times C_2$ if and only if d has one of the 34 values given in [6, Proposition 11.1(ii)].

Theorem 4.1. *Let $b \in \{7, 11, 19, 31, 59\}$, and set $A_1 = [1, 0, 3b]$, $A_2 = [2, 2, (3b+1)/2]$, $A_3 = [3, 0, b]$ and $A_6 = [6, 6, (b+3)/2]$. Let $i \in \{1, 2, 3, 6\}$, $n \in \mathbb{N}$ and $in = 2^{\alpha_i} 3^{\beta_i} n_0$ with $2 \nmid n_0$ and $3 \nmid n_0$. Then $R(A_i, n) > 0$ if and only if $2 \mid \operatorname{ord}_q n_0$ for every prime q with $(\frac{-3b}{q}) = -1$ and*

$$n_0 \equiv \begin{cases} 1 \pmod{12} & \text{if } 2 \mid \alpha_i \text{ and } 2 \mid \beta_i, \\ b \pmod{12} & \text{if } 2 \mid \alpha_i \text{ and } 2 \nmid \beta_i, \\ (3b+1)/2 \pmod{12} & \text{if } 2 \nmid \alpha_i \text{ and } 2 \mid \beta_i, \\ (b+3)/2 \pmod{12} & \text{if } 2 \nmid \alpha_i \text{ and } 2 \nmid \beta_i. \end{cases}$$

Moreover, if $R(A_i, n) > 0$, then $R(A_i, n) = 2 \prod_{(\frac{-3b}{p})=1} (1 + \operatorname{ord}_p n_0)$.

Proof. It is known that $f(-12b) = 1$ and $H(-12b) = \{A_1, A_2, A_3, A_6\} \cong C_2 \times C_2$. As $2 \nmid n_0$ and $3 \nmid n_0$, we see that

$$\begin{aligned} R(A_1, n_0) > 0 &\Rightarrow n_0 \equiv 1 \pmod{12}, \\ R(A_3, n_0) > 0 &\Rightarrow n_0 \equiv b \pmod{12}, \\ R(A_2, n_0) > 0 &\Rightarrow n_0 \equiv (3b+1)/2 \pmod{12}, \\ R(A_6, n_0) > 0 &\Rightarrow n_0 \equiv (b+3)/2 \pmod{12}. \end{aligned}$$

If $2 \nmid \operatorname{ord}_q n_0$ for some prime q with $(\frac{-3b}{q}) = -1$, we see that $q \mid n_0$, $2 \nmid \operatorname{ord}_q n$ and $(\frac{-12b}{q}) = -1$. Thus, applying Lemma 2.1 we have $N(n, -12b) = 0$ and so $R(A_i, n) = 0$.

Suppose that $2 \mid \operatorname{ord}_q n_0$ for every prime q with $(\frac{-3b}{q}) = -1$. From Lemma 2.1 we have $N(n_0, -12b) > 0$. Observe that $1, b, (3b+1)/2$ and $(b+3)/2$ are incongruent modulo 12. Applying the above we deduce

$$\begin{aligned}R(A_1,n_0)>0 &\Leftrightarrow n_0\equiv 1 \pmod{12},\\ R(A_3,n_0)>0 &\Leftrightarrow n_0\equiv b \pmod{12},\\ R(A_2,n_0)>0 &\Leftrightarrow n_0\equiv (3b+1)/2 \pmod{12},\\ R(A_6,n_0)>0 &\Leftrightarrow n_0\equiv (b+3)/2 \pmod{12}.\end{aligned}$$

Set

$$k_i=2^{\frac{1-(-1)^{\alpha_i}}{2}}3^{\frac{1-(-1)^{\beta_i}}{2}}=\begin{cases}1 & \text{if } 2\mid\alpha_i \text{ and } 2\mid\beta_i,\\ 2 & \text{if } 2\nmid\alpha_i \text{ and } 2\mid\beta_i,\\ 3 & \text{if } 2\mid\alpha_i \text{ and } 2\nmid\beta_i,\\ 6 & \text{if } 2\nmid\alpha_i \text{ and } 2\nmid\beta_i.\end{cases}$$

By Lemma 3.3 we have $\varphi_{k_i,1}(A_1)=A_{k_i}$. Thus applying Theorems 3.2 and 3.3 we get

$$R(A_i,n)=R(A_1,in)=R(\varphi_{k_i,1}(A_1),n_0)=R(A_{k_i},n_0).$$

Hence, using the above we deduce

$$\begin{aligned}R(A_i,n)>0 &\Leftrightarrow R(A_{k_i},n_0)>0\\ &\Leftrightarrow n_0\equiv\begin{cases}1 \pmod{12} & \text{if } 2\mid\alpha_i \text{ and } 2\mid\beta_i,\\ b \pmod{12} & \text{if } 2\mid\alpha_i \text{ and } 2\nmid\beta_i,\\ (3b+1)/2 \pmod{12} & \text{if } 2\nmid\alpha_i \text{ and } 2\mid\beta_i,\\ (b+3)/2 \pmod{12} & \text{if } 2\nmid\alpha_i \text{ and } 2\nmid\beta_i.\end{cases}\end{aligned}$$

If $R(A_i,n)>0$, by Theorem 3.1 we have

$$R(A_i,n)=w(-12b)\prod_{(\frac{-12b}{p})=1}(1+\operatorname{ord}_p n)=2\prod_{(\frac{-3b}{p})=1}(1+\operatorname{ord}_p n_0).$$

So the theorem is proved.

In a similar way one can prove the following results.

THEOREM 4.2. *Let $m\in\{5,7,13,17\}$, $i\in\{1,2,3,6\}$, $n\in\mathbb{N}$ and $in=2^{\alpha_i}3^{\beta_i}n_0$ with $(6,n_0)=1$. Then $R([i,0,6m/i],n)>0$ if and only if $2\mid\operatorname{ord}_q n_0$ for every prime q with $(\frac{-6m}{q})=-1$ and*

$$n_0\equiv\begin{cases}1,6m+1 \pmod{24} & \text{if } 2\mid\alpha_i \text{ and } 2\mid\beta_i,\\ 2m+3,8m+3 \pmod{24} & \text{if } 2\mid\alpha_i \text{ and } 2\nmid\beta_i,\\ 3m+2,3m+8 \pmod{24} & \text{if } 2\nmid\alpha_i \text{ and } 2\mid\beta_i,\\ m,m+6 \pmod{24} & \text{if } 2\nmid\alpha_i \text{ and } 2\nmid\beta_i.\end{cases}$$

Moreover, if $R([i,0,6m/i],n)>0$, then

$$R([i,0,6m/i],n)=2\prod_{(\frac{-6m}{p})=1}(1+\operatorname{ord}_p n_0).$$

THEOREM 4.3. *Let* $m \in \{7,13,19\}$, $i \in \{1,2,5,10\}$, $n \in \mathbb{N}$ *and* $in = 2^{\alpha_i}5^{\beta_i}n_0$ *with* $(10,n_0)=1$. *Then* $R([i,0,10m/i],n)>0$ *if and only if* $2\mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-10m}{q})=-1$ *and*

$$n_0 \equiv \begin{cases} 1, 9, 1+10m, 9+10m \pmod{40} & \text{if } 2\mid\alpha_i \text{ and } 2\mid\beta_i,\\ 5+2m, 5+8m, 5+18m, 5+32m \pmod{40} & \text{if } 2\mid\alpha_i \text{ and } 2\nmid\beta_i,\\ 5m+2, 5m+8, 5m+18, 5m+32 \pmod{40} & \text{if } 2\nmid\alpha_i \text{ and } 2\mid\beta_i,\\ m, 9m, 10+m, 10+9m \pmod{40} & \text{if } 2\nmid\alpha_i \text{ and } 2\nmid\beta_i. \end{cases}$$

Moreover, if $R([i,0,10m/i],n)>0$, *then*

$$R([i,0,10m/i],n) = 2\prod_{(\frac{-10m}{p})=1}(1+\mathrm{ord}_p n_0).$$

THEOREM 4.4. *Let* $A_1=[1,0,85]$, $A_2=[2,2,43]$, $A_5=[5,0,17]$ *and* $A_{10}=[10,10,11]$. *Let* $i\in\{1,2,5,10\}$, $n\in\mathbb{N}$ *and* $in=2^{\alpha_i}5^{\beta_i}n_0$ *with* $(10,n_0)=1$. *Then* $R(A_i,n)>0$ *if and only if* $2\mid\mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-85}{q})=-1$ *and*

$$n_0 \equiv \begin{cases} 1, 9 \pmod{20} & \text{if } 2\mid\alpha_i \text{ and } 2\mid\beta_i,\\ 13, 17 \pmod{20} & \text{if } 2\mid\alpha_i \text{ and } 2\nmid\beta_i,\\ 3, 7 \pmod{20} & \text{if } 2\nmid\alpha_i \text{ and } 2\mid\beta_i,\\ 11, 19 \pmod{20} & \text{if } 2\nmid\alpha_i \text{ and } 2\nmid\beta_i. \end{cases}$$

Moreover, if $R(A_i,n)>0$, *then* $R(A_i,n)=2\prod_{(\frac{-85}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 4.5. *Let* $A_1=[1,0,133]$, $A_2=[2,2,67]$, $A_7=[7,0,19]$ *and* $A_{14}=[14,14,13]=[13,12,13]$. *Let* $i\in\{1,2,7,14\}$, $n\in\mathbb{N}$ *and* $in=2^{\alpha_i}7^{\beta_i}n_0$ *with* $(14,n_0)=1$. *Then* $R(A_i,n)>0$ *if and only if* $2\mid\mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-133}{q})=-1$ *and*

$$n_0 \equiv \begin{cases} 1, 9, 25 \pmod{28} & \text{if } 2\mid\alpha_i \text{ and } 2\mid\beta_i,\\ 3, 19, 27 \pmod{28} & \text{if } 2\mid\alpha_i \text{ and } 2\nmid\beta_i,\\ 11, 15, 23 \pmod{28} & \text{if } 2\nmid\alpha_i \text{ and } 2\mid\beta_i,\\ 5, 13, 17 \pmod{28} & \text{if } 2\nmid\alpha_i \text{ and } 2\nmid\beta_i. \end{cases}$$

Moreover, if $R(A_i,n)>0$, *then* $R(A_i,n)=2\prod_{(\frac{-133}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 4.6. *Let* $A_1=[1,0,253]$, $A_2=[2,2,127]$, $A_{11}=[11,0,23]$ *and* $A_{22}=[22,22,17]=[17,12,17]$. *Let* $i\in\{1,2,11,22\}$, $n\in\mathbb{N}$ *and* $in=2^{\alpha_i}11^{\beta_i}n_0$ *with* $(22,n_0)=1$. *Then* $R(A_i,n)>0$ *if and only if* $2\mid\mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-253}{q})=-1$, $(\frac{-1}{n_0})=(-1)^{\alpha_i+\beta_i}$ *and* $(\frac{n_0}{11})=(-1)^{\alpha_i}$. *Moreover, if* $R(A_i,n)>0$, *then* $R(A_i,n)=2\prod_{(\frac{-253}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 4.7. *Let* $m\in\{13,29,37,53\}$, $i\in\{1,3,5,15\}$, $n\in\mathbb{N}$ *and* $in=3^{\alpha_i}5^{\beta_i}n_0$ *with* $(15,n_0)=1$. *Then* $R([i,i,(i+15m/i)/4],n)>0$ *if and only* *if* $2 \mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-15m}{q}) = -1$, $(\frac{n_0}{3}) = (-1)^{[(m-2)/3]\alpha_i+\beta_i}$ *and* $(\frac{n_0}{5}) = (-1)^{\alpha_i+[(m-3)/5]\beta_i}$. *Moreover, if* $R([i,i,(i+15m/i)/4],n) > 0$, *then* $R([i,i,(i+15m/i)/4],n) = 2\prod_{(\frac{-15m}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 4.8. *Let* $i \in \{1,3,7,21\}$, $n \in \mathbb{N}$ *and* $in = 3^{\alpha_i}7^{\beta_i}n_0$ *with* $(21,n_0) = 1$. *Then* $R([i,i,(i+483/i)/4],n) > 0$ *if and only if* $2 \mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-483}{q}) = -1$, $(\frac{n_0}{3}) = (-1)^{\alpha_i}$ *and* $(\frac{n_0}{7}) = (-1)^{\alpha_i+\beta_i}$. *Moreover, if* $R([i,i,(i+483/i)/4],n) > 0$, *then* $R([i,i,(i+483/i)/4],n) = 2\prod_{(\frac{-483}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 4.9. *Let* $i \in \{1,5,7,35\}$, $n \in \mathbb{N}$ *and* $in = 5^{\alpha_i}7^{\beta_i}n_0$ *with* $(35,n_0) = 1$. *Then* $R([i,i,(i+595/i)/4],n) > 0$ *if and only if* $2 \mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-595}{q}) = -1$, $(\frac{n_0}{5}) = (-1)^{\beta_i}$ *and* $(\frac{n_0}{7}) = (-1)^{\alpha_i}$. *Moreover, if* $R([i,i,(i+595/i)/4],n) > 0$, *then* $R([i,i,(i+595/i)/4],n) = 2\prod_{(\frac{-595}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 4.10. *Let* $i \in \{1,3,11,33\}$, $n \in \mathbb{N}$ *and* $in = 3^{\alpha_i}11^{\beta_i}n_0$ *with* $(33,n_0) = 1$. *Then* $R([i,i,(i+627/i)/4],n) > 0$ *if and only if* $2 \mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-627}{q}) = -1$, $(\frac{n_0}{3}) = (-1)^{\alpha_i+\beta_i}$ *and* $(\frac{n_0}{11}) = (-1)^{\beta_i}$. *Moreover, if* $R([i,i,(i+627/i)/4],n) > 0$, *then* $R([i,i,(i+627/i)/4],n) = 2\prod_{(\frac{-627}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 4.11. *Let* $i \in \{1,5,11,55\}$, $n \in \mathbb{N}$ *and* $in = 5^{\alpha_i}11^{\beta_i}n_0$ *with* $(55,n_0) = 1$. *Then* $R([i,i,(i+715/i)/4],n) > 0$ *if and only if* $2 \mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-715}{q}) = -1$, $(\frac{n_0}{5}) = (-1)^{\alpha_i}$ *and* $(\frac{n_0}{11}) = (-1)^{\beta_i}$. *Moreover, if* $R([i,i,(i+715/i)/4],n) > 0$, *then* $R([i,i,(i+715/i)/4],n) = 2\prod_{(\frac{-715}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 4.12. *Let* $i \in \{1,5,7,35\}$, $n \in \mathbb{N}$ *and* $in = 5^{\alpha_i}7^{\beta_i}n_0$ *with* $(35,n_0) = 1$. *Then* $R([i,i,(i+1435/i)/4],n) > 0$ *if and only if* $2 \mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-1435}{q}) = -1$, $(\frac{n_0}{5}) = (-1)^{\alpha_i+\beta_i}$ *and* $(\frac{n_0}{7}) = (-1)^{\alpha_i}$. *Moreover, if* $R([i,i,(i+1435/i)/4],n) > 0$, *then* $R([i,i,(i+1435/i)/4],n) = 2\prod_{(\frac{-1435}{p})=1}(1+\mathrm{ord}_p n_0)$.

5. Formulas for $t_n(a,b)$ when $4 \nmid a+b$ and $H(-4ab) \cong C_2 \times C_2$

THEOREM 5.1. *Let* $n \in \mathbb{N}$, $a \in \{1,3\}$, $b \in \{7,11,19,31,59\}$ *and* $4n + (a+3b/a)/2 = 3^{\beta}n_0$ $(3 \nmid n_0)$. *Then* $t_n(a,3b/a) > 0$ *if and only if* $2 \mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-3b}{q}) = -1$ *and*

$$n_0 \equiv \begin{cases} 1 \pmod 3 & \text{if } b \in \{11,59\} \text{ and } \beta \equiv (a+1)/2 \pmod 2, \\ 2 \pmod 3 & \text{otherwise.} \end{cases}$$

Moreover, if $t_n(a,3b/a) > 0$, *then* $t_n(a,3b/a) = \frac{1}{2}\prod_{(\frac{-3b}{p})=1}(1+\mathrm{ord}_p n_0)$.

Proof. As $b \equiv 3 \pmod 4$, we have $a + 3b/a \equiv 2 \pmod 4$. Thus, by Theorem 2.3 we have $4t_n(a, 3b/a) = R([a, 0, 3b/a], 8n + a + 3b/a)$. Since $a(8n + a + 3b/a) = 2 \cdot 3^{\beta+(a-1)/2} n_0$, applying Theorem 4.1 we deduce that $t_n(a, 3b/a) > 0$ if and only if $2 \mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{-3b}{q}) = -1$ and

$$(5.1)\qquad n_0 \equiv \begin{cases} (3b+1)/2 \pmod{12} & \text{if } 2 \mid \beta + (a-1)/2, \\ (b+3)/2 \pmod{12} & \text{if } 2 \nmid \beta + (a-1)/2. \end{cases}$$

From $4n + (a + 3b/a)/2 = 3^\beta n_0$ we know that $n_0 \equiv (3b + 1)/2$ or $(b+3)/2 \pmod 4$ according as $2 \mid \beta + (a-1)/2$ or $2 \nmid \beta + (a-1)/2$. Thus, (5.1) is equivalent to

$$\begin{aligned} n_0 &\equiv \begin{cases} (3b+1)/2 \equiv 2 \pmod 3 & \text{if } 2 \mid \beta + (a-1)/2, \\ (b+3)/2 \equiv 2b \pmod 3 & \text{if } 2 \nmid \beta + (a-1)/2 \end{cases} \\ &\equiv \begin{cases} 1 \pmod 3 & \text{if } b \in \{11, 59\} \text{ and } \beta \equiv (a+1)/2 \pmod 2, \\ 2 \pmod 3 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the result follows from Theorem 4.1.

From Theorems 4.2–4.6 and 2.3 one can similarly deduce the following results.

THEOREM 5.2. *Let $n \in \mathbb{N}$, $m \in \{5, 7, 13, 17\}$ and $a \in \{1, 2, 3, 6\}$. If $8n+a+6m/a = 3^\beta n_0$ $(3 \nmid n_0)$, then $t_n(a, 6m/a) > 0$ if and only if $2 \mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{-6m}{q}) = -1$ and*

$$n_0 \equiv \begin{cases} (-1)^{a+1} \pmod 3 & \text{if } m \in \{5, 17\}, \\ (-1)^\beta \mu(a) \pmod 3 & \text{if } m \in \{7, 13\}. \end{cases}$$

Moreover, if $t_n(a, 6m/a) > 0$, then $t_n(a, 6m/a) = \frac{1}{2} \prod_{(\frac{-6m}{p})=1} (1 + \mathrm{ord}_p n_0)$.

THEOREM 5.3. *Let $n \in \mathbb{N}$, $m \in \{7, 13, 19\}$, $a \in \{1, 2, 5, 10\}$ and $8n+a+10m/a = 5^\beta n_0$ $(5 \nmid n_0)$. Then $t_n(a, 10m/a) > 0$ if and only if $2 \mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{-10m}{q}) = -1$ and*

$$\left(\frac{n_0}{5}\right) = \begin{cases} (-1)^{a+1} & \text{if } m = 7, 13, \\ (-1)^\beta \mu(a) & \text{if } m = 19. \end{cases}$$

Moreover, if $t_n(a, 10m/a) > 0$, then $t_n(a, 10m/a) = \frac{1}{2} \prod_{(\frac{-10m}{p})=1} (1+\mathrm{ord}_p n_0)$.

THEOREM 5.4. *Let $n \in \mathbb{N}$, $a \in \{1, 5\}$ and $4n + (a + 85/a)/2 = 5^\beta n_0$ $(5 \nmid n_0)$. Then $t_n(a, 85/a) > 0$ if and only if $2 \mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{-85}{q}) = -1$ and $(\frac{n_0}{5}) = (-1)^{\beta+1}\mu(a)$. Moreover, if $t_n(a, 85/a) > 0$, then $t_n(a, 85/a) = \frac{1}{2} \prod_{(\frac{-85}{p})=1} (1 + \mathrm{ord}_p n_0)$.*

THEOREM 5.5. *Let $n \in \mathbb{N}$, $a \in \{1, 7\}$ and $4n + (a + 133/a)/2 = 7^\beta n_0$ $(7 \nmid n_0)$. Then $t_n(a, 133/a) > 0$ if and only if $2 \mid \mathrm{ord}_q n_0$ for every prime q*

with $(\frac{-133}{q}) = -1$ *and* $(\frac{n_0}{7}) = (-1)^{\beta}\mu(a)$. *Moreover, if* $t_n(a, 133/a) > 0$, *then* $t_n(a, 133/a) = \frac{1}{2}\prod_{(\frac{-133}{p})=1}(1 + \operatorname{ord}_p n_0)$.

THEOREM 5.6. *Let* $n \in \mathbb{N}$, $a \in \{1, 11\}$ *and* $4n + (a + 253/a)/2 = 11^{\beta} n_0$ $(11 \nmid n_0)$. *Then* $t_n(a, 253/a) > 0$ *if and only if* $2 \mid \operatorname{ord}_q n_0$ *for every prime* q *with* $(\frac{-253}{q}) = -1$ *and* $n_0 \equiv 2, 6, 7, 8, 10 \pmod{11}$. *Moreover, if* $t_n(a, 253/a) > 0$, *then* $t_n(a, 253/a) = \frac{1}{2}\prod_{(\frac{-253}{p})=1}(1 + \operatorname{ord}_p n_0)$.

6. Formulas for $R(K, n)$ when $K \in H(d)$, $d < 0$, $f(d) = 1$ and $H(d) \cong C_2 \times C_2 \times C_2$

THEOREM 6.1. *Let* $i, n \in \mathbb{N}$, $i \mid 30$ *and* $in = 2^{\alpha_i}3^{\beta_i}5^{\gamma_i}n_0$ *with* $(n_0, 30) = 1$. *Let* $m \in \{7, 11, 23\}$ *and*

$$A_i = \begin{cases} [i, 0, 15m/i] & \text{if } 2 \nmid i, \\ \left[i, i, \frac{1}{2}\left(\frac{i}{2} + \frac{15m}{i/2}\right)\right] & \text{if } 2 \mid i. \end{cases}$$

Then $R(A_i, n) > 0$ *if and only if* $2 \mid \operatorname{ord}_q n_0$ *for every prime* q *with* $(\frac{-15m}{q}) = -1$, $(\frac{-1}{n_0}) = (-1)^{((m+1)/4)\alpha_i + \beta_i}$, $(\frac{n_0}{3}) = (-1)^{\alpha_i + [(m+2)/3]\beta_i + \gamma_i}$ *and* $(\frac{n_0}{5}) = (-1)^{\alpha_i + \beta_i + [(m-3)/5]\gamma_i}$. *Moreover, if* $R(A_i, n) > 0$, *then* $R(A_i, n) = 2\prod_{(\frac{-15m}{p})=1}(1 + \operatorname{ord}_p n_0)$.

Proof. It is known that $f(-60m) = 1$ and $H(-60m) = \{A_k : k \in \{1, 2, 3, 5, 6, 10, 15, 30\}\} \cong C_2 \times C_2 \times C_2$. If $2 \nmid \operatorname{ord}_q n_0$ for some prime q with $(\frac{-15m}{q}) = -1$, then $2 \nmid \operatorname{ord}_q n$ and $(\frac{-60m}{q}) = -1$. Thus, applying Lemma 2.1 we have $N(n, -60m) = 2\delta(n, -60m) = 0$ and so $R(A_i, n) = 0$.

Suppose that $2 \mid \operatorname{ord}_q n_0$ for every prime q with $(\frac{-15m}{q}) = -1$. From Lemma 2.1 we have $N(n_0, -60m) > 0$. Now it is easily seen that $R(A_k, n_0) > 0$ depends only on the values of $(\frac{-1}{n_0})$, $(\frac{n_0}{3})$ and $(\frac{n_0}{5})$ given by Table 6.1.

Table 6.1. Criteria for $R(A_k, n_0) > 0$

k	A_k	$(\frac{-1}{n_0})$	$(\frac{n_0}{3})$	$(\frac{n_0}{5})$
1	$[1, 0, 15m]$	1	1	1
2	$[2, 2, \frac{1+15m}{2}]$	$(-1)^{(m+1)/4}$	-1	-1
3	$[3, 0, 5m]$	-1	$(-1)^{[(m+2)/3]}$	-1
5	$[5, 0, 3m]$	1	-1	$(-1)^{[(m-3)/5]}$
6	$[6, 6, \frac{3+5m}{2}]$	$-(-1)^{(m+1)/4}$	$-(-1)^{[(m+2)/3]}$	1
10	$[10, 10, \frac{5+3m}{2}]$	$(-1)^{(m+1)/4}$	1	$-(-1)^{[(m-3)/5]}$
15	$[15, 0, m]$	-1	$-(-1)^{[(m+2)/3]}$	$-(-1)^{[(m-3)/5]}$
30	$[30, 30, \frac{15+m}{2}]$	$-(-1)^{(m+1)/4}$	$(-1)^{[(m+2)/3]}$	$(-1)^{[(m-3)/5]}$

Set $k_i = 2^{\frac{1-(-1)^{\alpha_i}}{2}}3^{\frac{1-(-1)^{\beta_i}}{2}}5^{\frac{1-(-1)^{\gamma_i}}{2}}$. By Lemma 3.3 we have $\varphi_{k_i,1}(A_1) = A_{k_i}$. Thus applying Theorems 3.2 and 3.3 we get

$$R(A_i,n) = R(A_1,in) = R(\varphi_{k_i,1}(A_1),n_0) = R(A_{k_i},n_0).$$

Hence, using the above we deduce

$$\begin{aligned} R(A_i,n)>0 &\Leftrightarrow R(A_{k_i},n_0)>0 \\ &\Leftrightarrow \left(\frac{-1}{n_0}\right) = (-1)^{((m+1)/4)\alpha_i+\beta_i},\ \left(\frac{n_0}{3}\right) = (-1)^{\alpha_i+[(m+2)/3]\beta_i+\gamma_i} \\ &\qquad \text{and } \left(\frac{n_0}{5}\right) = (-1)^{\alpha_i+\beta_i+[(m-3)/5]\gamma_i}. \end{aligned}$$

If $R(A_i,n)>0$, by Theorem 3.1 we have

$$R(A_i,n) = w(-60m)\prod_{(\frac{-60m}{p})=1}(1+\operatorname{ord}_p n) = 2\prod_{(\frac{-15m}{p})=1}(1+\operatorname{ord}_p n_0).$$

So the theorem is proved.

In a similar way one can prove the following results.

THEOREM 6.2. *Let* $i,n\in\mathbb{N}$, $i\,|\,42$ *and* $in=2^{\alpha_i}3^{\beta_i}7^{\gamma_i}n_0$ *with* $(n_0,42)=1$. *Let*

$$A_i = \begin{cases} [i,0,273/i] & \text{if } 2\nmid i, \\ [i,i,(i^2+1092)/(4i)] & \text{if } 2\,|\,i. \end{cases}$$

Then $R(A_i,n)>0$ *if and only if* $2\,|\operatorname{ord}_q n_0$ *for every prime* q *with* $(\frac{-273}{q}) = -1$, $(\frac{-1}{n_0}) = (-1)^{\beta_i+\gamma_i}$, $(\frac{n_0}{3}) = (-1)^{\alpha_i}$ *and* $(\frac{n_0}{7}) = (-1)^{\beta_i}$. *Moreover, if* $R(A_i,n)>0$, *then* $R(A_i,n) = 2\prod_{(\frac{-273}{p})=1}(1+\operatorname{ord}_p n_0)$.

THEOREM 6.3. *Let* $i,n\in\mathbb{N}$, $i\,|\,42$ *and* $in=2^{\alpha_i}3^{\beta_i}7^{\gamma_i}n_0$ *with* $(n_0,42)=1$. *Let*

$$A_i = \begin{cases} [i,0,357/i] & \text{if } 2\nmid i, \\ [i,i,(i^2+1428)/(4i)] & \text{if } 2\,|\,i. \end{cases}$$

Then $R(A_i,n)>0$ *if and only if* $2\,|\operatorname{ord}_q n_0$ *for every prime* q *with* $(\frac{-357}{q}) = -1$, $(\frac{-1}{n_0}) = (-1)^{\alpha_i+\beta_i+\gamma_i}$, $(\frac{n_0}{3}) = (-1)^{\alpha_i+\beta_i}$ *and* $(\frac{n_0}{7}) = (-1)^{\beta_i}$. *Moreover, if* $R(A_i,n)>0$, *then* $R(A_i,n) = 2\prod_{(\frac{-357}{p})=1}(1+\operatorname{ord}_p n_0)$.

THEOREM 6.4. *Let* $i,n\in\mathbb{N}$, $i\,|\,70$ *and* $in=2^{\alpha_i}5^{\beta_i}7^{\gamma_i}n_0$ *with* $(n_0,70)=1$. *Let*

$$A_i = \begin{cases} [i,0,385/i] & \text{if } 2\nmid i, \\ [i,i,(i^2+1540)/(4i)] & \text{if } 2\,|\,i. \end{cases}$$

Then $R(A_i,n)>0$ *if and only if* $2\,|\operatorname{ord}_q n_0$ *for every prime* q *with* $(\frac{-385}{q}) = -1$, $(\frac{-1}{n_0}) = (-1)^{\gamma_i}$, $(\frac{n_0}{5}) = (-1)^{\alpha_i+\beta_i+\gamma_i}$ *and* $(\frac{n_0}{7}) = (-1)^{\beta_i+\gamma_i}$. *Moreover, if* $R(A_i,n)>0$, *then* $R(A_i,n) = 2\prod_{(\frac{-385}{p})=1}(1+\operatorname{ord}_p n_0)$.

THEOREM 6.5. *Let* $i,n\in\mathbb{N}$, $i\mid 30$ *and* $in=2^{\alpha_i}3^{\beta_i}5^{\gamma_i}n_0$ *with* $(n_0,30)=1$. *Then* $R([i,0,210/i],n)>0$ *if and only if* $2\mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-210}{q})=-1$, $(\frac{-2}{n_0})=(-1)^{\gamma_i}$, $(\frac{n_0}{3})=(-1)^{\alpha_i+\gamma_i}$ *and* $(\frac{n_0}{5})=(-1)^{\alpha_i+\beta_i+\gamma_i}$. *Moreover, if* $R([i,0,210/i],n)>0$, *then* $R([i,0,210/i],n)=2\prod_{(\frac{-210}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 6.6. *Let* $i,n\in\mathbb{N}$, $i\mid 30$ *and* $in=2^{\alpha_i}3^{\beta_i}5^{\gamma_i}n_0$ *with* $(n_0,30)=1$. *Then* $R([i,0,330/i],n)>0$ *if and only if* $2\mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-330}{q})=-1$, $(\frac{-2}{n_0})=(-1)^{\alpha_i+\gamma_i}$, $(\frac{n_0}{3})=(-1)^{\alpha_i+\beta_i+\gamma_i}$ *and* $(\frac{n_0}{5})=(-1)^{\alpha_i+\beta_i}$. *Moreover, if* $R([i,0,330/i],n)>0$, *then* $R([i,0,330/i],n)=2\prod_{(\frac{-330}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 6.7. *Let* $i,n\in\mathbb{N}$, $i\mid 42$ *and* $in=2^{\alpha_i}3^{\beta_i}7^{\gamma_i}n_0$ *with* $(n_0,42)=1$. *Then* $R([i,0,462/i],n)>0$ *if and only if* $2\mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-462}{q})=-1$, $(\frac{2}{n_0})=(-1)^{\beta_i}$, $(\frac{n_0}{3})=(-1)^{\alpha_i}$ *and* $(\frac{n_0}{7})=(-1)^{\beta_i+\gamma_i}$. *Moreover, if* $R([i,0,462/i],n)>0$, *then* $R([i,0,462/i],n)=2\prod_{(\frac{-462}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 6.8. *Let* $i,n\in\mathbb{N}$, $i\mid 105$ *and* $in=3^{\alpha_i}5^{\beta_i}7^{\gamma_i}n_0$ *with* $(n_0,105)=1$. *Then* $R([i,i,(i^2+1155)/(4i)],n)>0$ *if and only if* $2\mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-1155}{q})=-1$, $(\frac{n_0}{3})=(-1)^{\beta_i}$, $(\frac{n_0}{5})=(-1)^{\alpha_i+\gamma_i}$ *and* $(\frac{n_0}{7})=(-1)^{\alpha_i+\beta_i}$. *Moreover, if* $R([i,i,(i^2+1155)/(4i)],n)>0$, *then* $R([i,i,(i^2+1155)/(4i)],n)=2\prod_{(\frac{-1155}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 6.9. *Let* $i,n\in\mathbb{N}$, $i\mid 105$ *and* $in=3^{\alpha_i}5^{\beta_i}7^{\gamma_i}n_0$ *with* $(n_0,105)=1$. *Then* $R([i,i,(i^2+1995)/(4i)],n)>0$ *if and only if* $2\mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-1995}{q})=-1$, $(\frac{n_0}{3})=(-1)^{\alpha_i+\beta_i}$, $(\frac{n_0}{5})=(-1)^{\alpha_i+\gamma_i}$ *and* $(\frac{n_0}{7})=(-1)^{\alpha_i+\beta_i+\gamma_i}$. *Moreover, if* $R([i,i,(i^2+1995)/(4i)],n)>0$, *then* $R([i,i,(i^2+1995)/(4i)],n)=2\prod_{(\frac{-1995}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 6.10. *Let* $i,n\in\mathbb{N}$, $i\mid 231$ *and* $in=3^{\alpha_i}7^{\beta_i}11^{\gamma_i}n_0$ *with* $(n_0,231)=1$. *Then* $R([i,i,(i^2+3003)/(4i)],n)>0$ *if and only if* $2\mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-3003}{q})=-1$, $(\frac{n_0}{3})=(-1)^{\alpha_i+\gamma_i}$, $(\frac{n_0}{7})=(-1)^{\alpha_i}$ *and* $(\frac{n_0}{11})=(-1)^{\beta_i}$. *Moreover, if* $R([i,i,(i^2+3003)/(4i)],n)>0$, *then* $R([i,i,(i^2+3003)/(4i)],n)=2\prod_{(\frac{-3003}{p})=1}(1+\mathrm{ord}_p n_0)$.

THEOREM 6.11. *Let* $i,n\in\mathbb{N}$, $i\mid 195$ *and* $in=3^{\alpha_i}5^{\beta_i}13^{\gamma_i}n_0$ *with* $(n_0,195)=1$. *Then* $R([i,i,(i^2+3315)/(4i)],n)>0$ *if and only if* $2\mid \mathrm{ord}_q n_0$ *for every prime* q *with* $(\frac{-3315}{q})=-1$, $(\frac{n_0}{3})=(-1)^{\beta_i}$, $(\frac{n_0}{5})=(-1)^{\alpha_i+\beta_i+\gamma_i}$ *and* $(\frac{n_0}{13})=(-1)^{\beta_i+\gamma_i}$. *Moreover, if* $R([i,i,(i^2+3315)/(4i)],n)>0$, *then* $R([i,i,(i^2+3315)/(4i)],n)=2\prod_{(\frac{-3315}{p})=1}(1+\mathrm{ord}_p n_0)$.

7. Formulas for $t_n(a,b)$ when $4\nmid a+b$ and $H(-4ab)\cong C_2\times C_2\times C_2$

THEOREM 7.1. *Let $m\in\{7,11,23\}$, $a\in\{1,3,5,15\}$ and $n\in\mathbb{N}$. If $a(4n+(a+15m/a)/2)=3^\beta 5^\gamma n_0$ with $3\nmid n_0$ and $5\nmid n_0$, then $t_n(a,15m/a)>0$ if and only if $2\mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{-15m}{q})=-1$, $(\frac{n_0}{3})=-(-1)^{[(m+2)/3]\beta+\gamma}$ and $(\frac{n_0}{5})=-(-1)^{\beta+[(m-3)/5]\gamma}$. Moreover, if $t_n(a,15m/a)>0$, then $t_n(a,15m/a)=\frac12\prod_{(\frac{-15m}{p})=1}(1+\mathrm{ord}_p n_0)$.*

Proof. Since $(-1)^{(m+1)/4}\equiv(1-m)/2\equiv(a^2+15m)/2\equiv 4na+(a^2+15m)/2=3^\beta 5^\gamma n_0\equiv(-1)^\beta n_0 \pmod 4$, we have $(\frac{-1}{n_0})=(-1)^{(m+1)/4+\beta}$. As $a+15m/a\equiv a+a\equiv 2 \pmod 4$, by Theorem 2.3 we have $4t_n(a,15m/a)=R([a,0,15m/a],8n+a+15m/a)$. Now applying the above and Theorem 6.1 we deduce the result.

In a similar way, using Theorems 6.2–6.7 one can prove the following results.

THEOREM 7.2. *Let $n\in\mathbb{N}$ and $a\in\{1,3,7,21\}$. If $4n+(a+273/a)/2=3^\beta 7^\gamma n_0$ with $3\nmid n_0$ and $7\nmid n_0$, then $t_n(a,273/a)>0$ if and only if $2\mid\mathrm{ord}_q n_0$ for every prime q with $(\frac{-273}{q})=-1$, $n_0\equiv 2 \pmod 3$ and $(\frac{n_0}{7})=(-1)^\beta(\frac{2}{a})$. Moreover, if $t_n(a,273/a)>0$, then $t_n(a,273/a)=\frac12\prod_{(\frac{-273}{p})=1}(1+\mathrm{ord}_p n_0)$.*

THEOREM 7.3. *Let $n\in\mathbb{N}$ and $a\in\{1,3,7,21\}$. If $4n+(a+357/a)/2=3^\beta 7^\gamma n_0$ with $3\nmid n_0$ and $7\nmid n_0$, then $t_n(a,357/a)>0$ if and only if $2\mid\mathrm{ord}_q n_0$ for every prime q with $(\frac{-357}{q})=-1$ and $(\frac{n_0}{7})=-(\frac{n_0}{3})=(-1)^\beta(\frac{2}{a})$. Moreover, if $t_n(a,357/a)>0$, then $t_n(a,357/a)=\frac12\prod_{(\frac{-357}{p})=1}(1+\mathrm{ord}_p n_0)$.*

THEOREM 7.4. *Let $n\in\mathbb{N}$ and $a\in\{1,5,7,35\}$. If $4n+(a+385/a)/2=5^\beta 7^\gamma n_0$ with $5\nmid n_0$ and $7\nmid n_0$, then $t_n(a,385/a)>0$ if and only if $2\mid\mathrm{ord}_q n_0$ for every prime q with $(\frac{-385}{q})=-1$ and $(\frac{n_0}{7})=-(\frac{n_0}{5})=(-1)^{\beta+\gamma}\mu(a)$. Moreover, if $t_n(a,385/a)>0$, then $t_n(a,385/a)=\frac12\prod_{(\frac{-385}{p})=1}(1+\mathrm{ord}_p n_0)$.*

THEOREM 7.5. *Let $a,n\in\mathbb{N}$, $a\mid 30$ and $\frac{a}{(2,a)}(8n+a+210/a)=3^\beta 5^\gamma n_0$ with $3\nmid n_0$ and $5\nmid n_0$. Then $t_n(a,210/a)>0$ if and only if $2\mid\mathrm{ord}_q n_0$ for every prime q with $(\frac{-210}{q})=-1$ and $(\frac{n_0}{3})=(-1)^\beta(\frac{n_0}{5})=(-1)^{a-1+\gamma}$. Moreover, if $t_n(a,210/a)>0$, then $t_n(a,210/a)=\frac12\prod_{(\frac{-210}{p})=1}(1+\mathrm{ord}_p n_0)$.*

THEOREM 7.6. *Let $a,n\in\mathbb{N}$, $a\mid 30$ and $\frac{a}{(2,a)}(8n+a+330/a)=3^\beta 5^\gamma n_0$ with $3\nmid n_0$ and $5\nmid n_0$. Then $t_n(a,330/a)>0$ if and only if $2\mid\mathrm{ord}_q n_0$ for every prime q with $(\frac{-330}{q})=-1$ and $(\frac{n_0}{5})=(-1)^\gamma(\frac{n_0}{3})=(-1)^{a-1+\beta}$. Moreover, if $t_n(a,330/a)>0$, then $t_n(a,330/a)=\frac12\prod_{(\frac{-330}{p})=1}(1+\mathrm{ord}_p n_0)$.*

THEOREM 7.7. *Let $a,n\in\mathbb{N}$, $a\mid 42$ and $\frac{a}{(2,a)}(8n+a+462/a)=3^\beta 7^\gamma n_0$ with $3\nmid n_0$ and $7\nmid n_0$. Then $t_n(a,462/a)>0$ if and only if $2\mid\mathrm{ord}_q n_0$*

for every prime q with $(\frac{-462}{q}) = -1$, $(\frac{n_0}{3}) = (-1)^{a-1}$ and $(\frac{n_0}{7}) = (-1)^{\beta+\gamma}$. Moreover, if $t_n(a, 462/a) > 0$, then $t_n(a, 462/a) = \frac{1}{2}\prod_{(\frac{-462}{p})=1}(1+\text{ord}_p\, n_0)$.

8. Formulas for $t_n(a, 1365/a)$

THEOREM 8.1. *Let $i, n \in \mathbb{N}$, $i \mid 210$ and $in = 2^{\alpha_i}3^{\beta_i}5^{\gamma_i}7^{\delta_i}n_0$ with $(n_0, 210) = 1$. Set $A_i = [i, 0, 1365/i]$ or $[i, i, (i^2 + 5460)/(4i)]$ according as $2 \nmid i$ or $2 \mid i$. Then $R(A_i, n) > 0$ if and only if $2 \mid \text{ord}_q\, n_0$ for every prime q with $(\frac{-1365}{q}) = -1$, $(\frac{-1}{n_0}) = (-1)^{\alpha_i+\beta_i+\delta_i}$, $(\frac{n_0}{3}) = (-1)^{\alpha_i+\beta_i+\gamma_i}$, $(\frac{n_0}{5}) = (-1)^{\alpha_i+\beta_i+\gamma_i+\delta_i}$ and $(\frac{n_0}{7}) = (-1)^{\beta_i+\gamma_i+\delta_i}$. Moreover, if $R(A_i, n) > 0$, then $R(A_i, n) = 2\prod_{(\frac{-1365}{p})=1}(1 + \text{ord}_p\, n_0)$.*

Proof. It is known that $f(-5460) = 1$ and $H(-5460) = \{A_k : k \in \mathbb{N}, k \mid 210\} \cong C_2 \times C_2 \times C_2 \times C_2$. If $2 \nmid \text{ord}_q\, n_0$ for some prime q with $(\frac{-1365}{q}) = -1$, we see that $2 \nmid \text{ord}_q\, n$ and $(\frac{-5460}{q}) = -1$. Thus, applying Lemma 2.1 we have $N(n, -5460) = 2\delta(n, -5460) = 0$ and so $R(A_i, n) = 0$.

Suppose that $2 \mid \text{ord}_q\, n_0$ for every prime q with $(\frac{-1365}{q}) = -1$. From Lemma 2.1 we have $N(n_0, -5460) > 0$. Now it is easily seen that $R(A_i, n_0) > 0$ depends only on the values of $(\frac{-1}{n_0})$, $(\frac{n_0}{3})$, $(\frac{n_0}{5})$ and $(\frac{n_0}{7})$ given by the following table.

Table 8.1. Criteria for $R(A_i, n_0) > 0$

i	A_i	$(\frac{-1}{n_0})$	$(\frac{n_0}{3})$	$(\frac{n_0}{5})$	$(\frac{n_0}{7})$
1	$[1, 0, 1365]$	1	1	1	1
2	$[2, 2, 683]$	−1	−1	−1	1
3	$[3, 0, 455]$	−1	−1	−1	−1
5	$[5, 0, 273]$	1	−1	−1	−1
6	$[6, 6, 229]$	1	1	1	−1
7	$[7, 0, 195]$	−1	1	−1	−1
10	$[10, 10, 139]$	−1	1	1	−1
14	$[14, 14, 101]$	1	−1	1	−1
15	$[15, 0, 91]$	−1	1	1	1
21	$[21, 0, 65]$	1	−1	1	1
30	$[30, 30, 53]$	1	−1	−1	1
35	$[35, 0, 39]$	−1	−1	1	1
42	$[42, 42, 43]$	−1	1	−1	1
70	$[70, 70, 37]$	1	1	−1	1
105	$[105, 0, 13]$	1	1	−1	−1
210	$[210, 210, 59]$	−1	−1	1	−1

Set

$$k = 2^{\frac{1-(-1)^{\alpha_1}}{2}} 3^{\frac{1-(-1)^{\beta_1}}{2}} 5^{\frac{1-(-1)^{\gamma_1}}{2}} 7^{\frac{1-(-1)^{\delta_1}}{2}}.$$

By Lemma 3.3 we have $\varphi_{k,1}(A_1) = A_k$. Thus applying Theorem 3.2 we get

$$R(A_1, n) = R(\varphi_{k,1}(A_1), n_0) = R(A_k, n_0).$$

Hence, using the above we deduce

$$\begin{aligned} R(A_1, n) > 0 \ &\Leftrightarrow R(A_k, n_0) > 0 \\ &\Leftrightarrow \left(\frac{-1}{n_0}\right) = (-1)^{\alpha_1+\beta_1+\delta_1}, \left(\frac{n_0}{3}\right) = (-1)^{\alpha_1+\beta_1+\gamma_1}, \\ &\quad \left(\frac{n_0}{5}\right) = (-1)^{\alpha_1+\beta_1+\gamma_1+\delta_1} \text{ and } \left(\frac{n_0}{7}\right) = (-1)^{\beta_1+\gamma_1+\delta_1}. \end{aligned}$$

To see the criteria for $R(A_i, n) > 0$ $(i > 1)$, we note that $R(A_i, n) = R(A_1, in)$ by Theorem 3.3.

If $R(A_i, n) > 0$, by Theorem 3.1 we have

$$R(A_i, n) = w(-5460) \prod_{(\frac{-5460}{p})=1} (1 + \operatorname{ord}_p n) = 2 \prod_{(\frac{-1365}{p})=1} (1 + \operatorname{ord}_p n_0).$$

So the theorem is proved.

THEOREM 8.2. *Let $a, n \in \mathbb{N}$, $a \mid 105$ and $4an + (a^2 + 1365)/2 = 3^\beta 5^\gamma 7^\delta n_0$ with $(n_0, 105) = 1$. Then $t_n(a, 1365/a) > 0$ if and only if $2 \mid \operatorname{ord}_q n_0$ for every prime q with $(\frac{-1365}{q}) = -1$, $(\frac{n_0}{3}) = -(-1)^{\beta+\gamma}$ and $(\frac{n_0}{7}) = -(\frac{n_0}{5}) = (-1)^{\beta+\gamma+\delta}$. Moreover, if $t_n(a, 1365/a) > 0$, then $t_n(a, 1365/a) = \frac{1}{2} \prod_{(\frac{-1365}{p})=1} (1 + \operatorname{ord}_p n_0)$.*

Proof. From Theorem 2.3(i) we know that

$$4t_n(a, 1365/a) = R([a, 0, 1365/a], 8n + a + 1365/a).$$

Since $4an + (a^2 + 1365)/2 = 3^\beta 5^\gamma 7^\delta n_0$ we see that $(-1)^{\beta+\delta} n_0 \equiv 3^\beta 5^\gamma 7^\delta n_0 \equiv (a^2 + 1365)/2 \equiv 3 \pmod 4$ and so $(\frac{-1}{n_0}) = -(-1)^{\beta+\delta}$. As $a(8n + a + 1365/a) = 2 \cdot 3^\beta 5^\gamma 7^\delta n_0$, using Theorem 8.1 and the above we get

$$\begin{aligned} &t_n(a, 1365/a) > 0 \\ &\quad \Leftrightarrow R([a, 0, 1365/a], 8n + a + 1365/a) > 0 \\ &\quad \Leftrightarrow \left(\frac{n_0}{3}\right) = -(-1)^{\beta+\gamma}, \left(\frac{n_0}{5}\right) = -(-1)^{\beta+\gamma+\delta}, \left(\frac{n_0}{7}\right) = (-1)^{\beta+\gamma+\delta}. \end{aligned}$$

When $t_n(a, 1365/a) > 0$, using Theorem 8.1 we deduce the remaining result.

9. Formulas for $R(A, n)$ when $d < 0$ and $H(d) = \{I, A, A^2, A^3\}$. From [6, Proposition 11.1] we know that all negative discriminants d with

$H(d) \cong C_4$ are:

$$-39, -55, -56, -63, -68, -80, -128, -136, -144, -155, -156, -171, -184, -196, -203, -208, -219, -220, -252, -256, -259, -275, -291, -292, -323, -328, -355, -363, -387, -388, -400, -475, -507, -568, -592, -603, -667, -723, -763, -772, -955, -1003, -1027, -1227, -1243, -1387, -1411, -1467, -1507, -1555.$$

From [6, Theorem 11.3] and Lemma 2.1 we deduce the following result.

LEMMA 9.1. *Let d be a discriminant with conductor f. Suppose $H(d) = \{I, A, A^2, A^3\}$ with $A^4 = I$ and $n \in \mathbb{N}$. Then*

$$R(A,n) = \begin{cases} 0 & \text{if } (n,f^2) \text{ is not a square,} \\ N(n,d)/4 & \text{if } (n,f^2) = m^2 \text{ with } m \in \mathbb{N} \text{ and } h(d/m^2) = 1, \\ (1-(-1)^{\sum_{p\in R(A_0)} \operatorname{ord}_p n})N(n,d)/4 & \\ & \text{if } (n,f^2) = m^2 \text{ with } m \in \mathbb{N} \text{ and } h(d/m^2) > 1, \end{cases}$$

where A_0 is a generator of $H(d/m^2)$. Hence $R(A,n) = 0$, $N(n,d)/4$ or $N(n,d)/2$.

LEMMA 9.2. *Let d be a discriminant such that $H(d) = \{I, A, A^2, A^3\}$ with $A^4 = I$. Then no prime divisor of d can be represented by A.*

Proof. Suppose that p is a prime divisor of d and $f = f(d)$. If $p \mid f$, by [6, Lemma 5.2(i)] we know that p is not represented by any class in $H(d)$. If $p \nmid f$, by [6, Lemma 5.2(ii)] we know that p is represented by exactly one class $K \in H(d)$ and $K = K^{-1}$. Thus p is not represented by A. This proves the lemma.

LEMMA 9.3. *Let d be a discriminant and $a, b, c \in \mathbb{Z}$ with $b^2 - 4ac = d$. Let p be a prime such that $p = ax^2 + bxy + cy^2$ for some $x, y \in \mathbb{Z}$. Let q be an odd prime such that $q \mid d$ and $q \nmid ap$. Then $(\frac{p}{q}) = (\frac{a}{q})$.*

Proof. As $4ap = (2ax + by)^2 - dy^2$ we obtain the result.

LEMMA 9.4. *Let d be a discriminant with conductor f and $d_0 = d/f^2$. Let $H(d) = \{I, A, A^2, A^3\}$ with $A^4 = I$, $n \in \mathbb{N}$ and $(n, f) = 1$. Suppose that q is an odd prime divisor of d such that for any prime $p \neq q$,*

$$p \in R(I) \cup R(A^2) \ \Rightarrow \left(\frac{p}{q}\right) = 1, \quad \text{and} \quad p \in R(A) \ \Rightarrow \left(\frac{p}{q}\right) = -1.$$

Suppose $n = q^\alpha n_0$ ($q \nmid n_0$). Then

$$R(A,n) = \left(1 - \left(\frac{n_0}{q}\right)\right) \frac{w(d)}{4} \sum_{k \mid n} \left(\frac{d_0}{k}\right).$$

Proof. As $(n,f)=1$, by Lemmas 2.1 and 9.1 we have

$$\begin{aligned}R(A,n)&=(1-(-1)^{\sum_{p\in R(A)}\mathrm{ord}_p n})\frac{N(n,d)}{4}\\&=(1-(-1)^{\sum_{p\in R(A)}\mathrm{ord}_p n})\frac{w(d)}{4}\sum_{k\mid n}\Big(\frac{d_0}{k}\Big).\end{aligned}$$

If there is a prime p such that $(\frac{d}{p})=-1$ and $2\nmid \mathrm{ord}_p n$, by (2.1) we have $\sum_{k\mid n}(\frac{d_0}{k})=\sum_{k\mid n}(\frac{d}{k})=0$ and so $R(A,n)=0$. Hence the result holds. Now assume $2\mid \mathrm{ord}_p n$ for every prime p with $(\frac{d}{p})=-1$. If p is a prime such that $p\mid d$ and $p\mid n$, as $(n,f)=1$ we have $p\nmid f$. From Lemma 9.2 and its proof we know that $p\in R(I)\cup R(A^2)$ and $p\notin R(A)$. Thus $q\notin R(A)$. Hence

$$n_0=\prod_{(\frac{d}{p})=-1}p^{\mathrm{ord}_p n}\prod_{p\in R(I)\cup R(A^2),\,p\ne q}p^{\mathrm{ord}_p n}\prod_{p\in R(A)}p^{\mathrm{ord}_p n}$$

and therefore

$$\begin{aligned}\Big(\frac{n_0}{q}\Big)&=\prod_{(\frac{d}{p})=-1}\Big(\frac{p}{q}\Big)^{\mathrm{ord}_p n}\prod_{p\in R(I)\cup R(A^2),\,p\ne q}\Big(\frac{p}{q}\Big)^{\mathrm{ord}_p n}\prod_{p\in R(A)}\Big(\frac{p}{q}\Big)^{\mathrm{ord}_p n}\\&=\prod_{p\in R(A)}(-1)^{\mathrm{ord}_p n}=(-1)^{\sum_{p\in R(A)}\mathrm{ord}_p n}.\end{aligned}$$

Now putting the above together we obtain the result.

LEMMA 9.5. *Let d be a discriminant with conductor f and $4\mid d$. Let $H(d)=\{I,A,A^2,A^3\}$ with $A^4=I$, $n\in\mathbb{N}$ and $(n,f)=1$. Suppose $a\in\{-1,2,-2\}$, $n=2^{\alpha}n_0$ $(2\nmid n_0)$ and for any odd prime p,*

$$p\in R(I)\cup R(A^2)\ \Rightarrow\ \Big(\frac{a}{p}\Big)=1,\quad \textit{and}\quad p\in R(A)\ \Rightarrow\ \Big(\frac{a}{p}\Big)=-1.$$

Then

$$R(A,n)=\Big(1-\Big(\frac{a}{n_0}\Big)\Big)\frac{w(d)}{4}\sum_{k\mid n}\Big(\frac{d/f^2}{k}\Big).$$

Proof. Replacing $q,(\frac{n_0}{q}),(\frac{p}{q})$ with $2,(\frac{a}{n_0}),(\frac{a}{p})$ in the proof of Lemma 9.4 we deduce the result.

LEMMA 9.6. *Let d be a negative discriminant with conductor f, $d_0=d/f^2$ and $H(d)=\{I,A,A^2,A^3\}$ with $A^4=I$. Let $n\in\mathbb{N}$ and $(n,f^2)=m^2$ for $m\in\mathbb{N}$. Suppose $h(d/m^2)=1$. Then*

$$R(A,n)=w\Big(\frac{d}{m^2}\Big)\sum_{k\mid n/m^2}\Big(\frac{d_0}{k}\Big).$$

Proof. From Lemmas 9.1 and 2.1 we have

$$R(A,n)=\frac{N(n,d)}{4}=\frac{1}{4}\cdot m\prod_{p\mid m}\Big(1-\frac{1}{p}\Big(\frac{d/m^2}{p}\Big)\Big)\cdot w(d)\sum_{k\mid n/m^2}\Big(\frac{d_0}{k}\Big).$$

As $h(d)=4$ and $h(d/m^2)=1$, by [6, Lemma 3.5] we have

$$m\prod_{p\mid m}\Big(1-\frac{1}{p}\Big(\frac{d/m^2}{p}\Big)\Big)\cdot w(d)=\frac{h(d)w(d/m^2)}{h(d/m^2)}=4w\Big(\frac{d}{m^2}\Big).$$

So the result follows.

Lemma 9.7. *Let d be a negative discriminant with conductor f, $d_0=d/f^2$ and $H(d)=\{I,A,A^2,A^3\}$ with $A^4=I$. Let $n\in\mathbb{N}$ and $(n,f^2)=m^2$ for $m\in\mathbb{N}$. Suppose $d/m^2\neq-60$ and $h(d/m^2)=2$. Then*

$$R(A,n)=\Big(1-\chi\Big(\frac{n}{m^2},\frac{d}{m^2}\Big)\Big)\sum_{k\mid n/m^2}\Big(\frac{d_0}{k}\Big),$$

where $\chi(n',d')\in\{1,-1\}$ is given by [6, Table 9.2].

Proof. For $K\in H(d)$, by [6, Lemma 2.1(ii)] we may assume $K=[a,bm,cm^2]$ with $(a,m)=1$. We recall that $\varphi_{1,m}([a,bm,cm^2])=[a,b,c]$. By [6, Theorem 2.1], $\varphi_{1,m}$ is a surjective homomorphism from $H(d)$ to $H(d/m^2)$. Suppose $H(d/m^2)=\{I_0,A_0\}$ with $A_0^2=I_0$. Then clearly $\varphi_{1,m}(A)=A_0$. Now applying [6, Theorem 3.2] we obtain $R(A,n)=R(A_0,n/m^2)$. As $d/m^2=d_0(f/m)^2$ and $(n/m^2,f/m)=1$, using (2.1) and [6, Theorem 9.3] we deduce $R(A_0,n/m^2)=(1-\chi(n/m^2,d/m^2))\sum_{k\mid n/m^2}(\frac{d_0}{k})$. So the result is true.

Theorem 9.1. *Let $n\in\mathbb{N}$. Then*

$$R([2,1,5],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{3}\Big)\Big)\sum_{k\mid n}\Big(\frac{-39}{k}\Big)\qquad(n=3^\alpha n_0,\ 3\nmid n_0),$$

$$R([2,1,7],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{5}\Big)\Big)\sum_{k\mid n}\Big(\frac{-55}{k}\Big)\qquad(n=5^\alpha n_0,\ 5\nmid n_0),$$

$$R([3,2,5],n)=\frac{1}{2}\Big(1-\Big(\frac{2}{n_0}\Big)\Big)\sum_{k\mid n}\Big(\frac{-56}{k}\Big)\qquad(n=2^\alpha n_0,\ 2\nmid n_0),$$

$$R([3,2,6],n)=\frac{1}{2}\Big(1-\Big(\frac{-1}{n_0}\Big)\Big)\sum_{k\mid n}\Big(\frac{-68}{k}\Big)\qquad(n=2^\alpha n_0,\ 2\nmid n_0),$$

$$R([5,2,7],n)=\frac{1}{2}\Big(1-\Big(\frac{-2}{n_0}\Big)\Big)\sum_{k\mid n}\Big(\frac{-136}{k}\Big)\qquad(n=2^\alpha n_0,\ 2\nmid n_0).$$

Proof. Clearly $H(-39)=\{[1,1,10],[2,1,5],[2,-1,5],[3,3,4]\}\cong C_4$ and $f(-39)=1$. Let $p\ne 3$ be a prime. If $p=x^2+xy+10y^2$ or $3x^2+3xy+4y^2$, by Lemma 9.3 we have $(\frac p3)=1$ and $(\frac p{13})=0,1$. If $p=2x^2+xy+5y^2$, by Lemmas 9.2 and 9.3 we have $(\frac p3)=(\frac p{13})=-1$. Thus taking $d=-39$, $f=1$, $A=[2,1,5]$ and $q=3$ in Lemma 9.4 we deduce the formula for $R([2,1,5],n)$.

It is known that $H(-55)=\{[1,1,14],[2,1,7],[2,-1,7],[4,3,4]\}\cong C_4$ and $f(-55)=1$. Let $p\ne 5$ be a prime. If $p=x^2+xy+14y^2$ or $4x^2+3xy+4y^2$, by Lemma 9.3 we have $(\frac p5)=1$ and $(\frac p{11})=0,1$. If $p=2x^2+xy+7y^2$, by Lemmas 9.2 and 9.3 we have $(\frac p5)=(\frac p{11})=-1$. Thus taking $d=-55$, $f=1$, $A=[2,1,7]$ and $q=5$ in Lemma 9.4 we deduce the formula for $R([2,1,7],n)$.

It is clear that $H(-56)=\{[1,0,14],[2,0,7],[3,2,5],[3,-2,5]\}\cong C_4$ and $f(-56)=1$. Let p be an odd prime. If $p=x^2+14y^2$ or $2x^2+7y^2$, then clearly $(\frac 2p)=1$ and $(\frac{-7}p)=0,1$. If $p=3x^2+2xy+5y^2$, then clearly $(\frac 2p)=(\frac{-7}p)=-1$. Hence taking $d=-56$, $f=1$, $A=[3,2,5]$ and $a=2$ in Lemma 9.5 we deduce the result for $R([3,2,5],n)$.

Clearly $H(-68)=\{[1,0,17],[2,2,9],[3,2,6],[3,-2,6]\}\cong C_4$ and $f(-68)=1$. Let p be an odd prime. If $p=x^2+17y^2$ or $2x^2+2xy+9y^2$, then clearly $(\frac{-1}p)=1$ and $(\frac{17}p)=0,1$. If $p=3x^2+2xy+6y^2$, then clearly $(\frac{-1}p)=-1$ and $(\frac{17}p)=-1$. Hence taking $d=-68$, $f=1$, $A=[3,2,6]$ and $a=-1$ in Lemma 9.5 we deduce the result for $R([3,2,6],n)$.

It is clear that $H(-136)=\{[1,0,34],[2,0,17],[5,2,7],[5,-2,7]\}\cong C_4$ and $f(-136)=1$. Let p be an odd prime. If $p=x^2+34y^2$ or $2x^2+17y^2$, then clearly $(\frac{-2}p)=1$ and $(\frac{17}p)=0,1$. If $p=5x^2+2xy+7y^2$, then clearly $(\frac{-2}p)=(\frac{17}p)=-1$. Hence taking $d=-136$, $f=1$, $A=[5,2,7]$ and $a=-2$ in Lemma 9.5 we deduce the result for $R([5,2,7],n)$.

By the above, the theorem is proved.

Using Lemmas 9.4 and 9.5 one can similarly prove the following results.

Theorem 9.2. *Let $n\in\mathbb N$. Then*

$$R([3,1,13],n)=\frac12\Big(1-\Big(\frac{n_0}{5}\Big)\Big)\sum_{k\mid n}\Big(\frac{-155}{k}\Big)\qquad (n=5^\alpha n_0,\ 5\nmid n_0),$$

$$R([5,4,10],n)=\frac12\Big(1-\Big(\frac{2}{n_0}\Big)\Big)\sum_{k\mid n}\Big(\frac{-184}{k}\Big)\qquad (n=2^\alpha n_0,\ 2\nmid n_0),$$

$$R([3,1,17],n)=\frac12\Big(1-\Big(\frac{n_0}{7}\Big)\Big)\sum_{k\mid n}\Big(\frac{-203}{k}\Big)\qquad (n=7^\alpha n_0,\ 7\nmid n_0),$$

$$R([5,1,11],n)=\frac12\Big(1-\Big(\frac{n_0}{3}\Big)\Big)\sum_{k\mid n}\Big(\frac{-219}{k}\Big)\qquad (n=3^\alpha n_0,\ 3\nmid n_0).$$

THEOREM 9.3. *Let $n \in \mathbb{N}$. Then*

$$R([5,1,13],n) = \frac{1}{2}\Big(1-\Big(\frac{n_0}{7}\Big)\Big)\sum_{k\mid n}\Big(\frac{-259}{k}\Big) \qquad (n = 7^{\alpha}n_0,\ 7 \nmid n_0),$$

$$R([5,3,15],n) = \frac{1}{2}\Big(1-\Big(\frac{n_0}{3}\Big)\Big)\sum_{k\mid n}\Big(\frac{-291}{k}\Big) \qquad (n = 3^{\alpha}n_0,\ 3 \nmid n_0),$$

$$R([7,4,11],n) = \frac{1}{2}\Big(1-\Big(\frac{-1}{n_0}\Big)\Big)\sum_{k\mid n}\Big(\frac{-292}{k}\Big) \qquad (n = 2^{\alpha}n_0,\ 2 \nmid n_0),$$

$$R([3,1,27],n) = \frac{1}{2}\Big(1-\Big(\frac{n_0}{17}\Big)\Big)\sum_{k\mid n}\Big(\frac{-323}{k}\Big) \qquad (n = 17^{\alpha}n_0,\ 17 \nmid n_0),$$

$$R([7,6,13],n) = \frac{1}{2}\Big(1-\Big(\frac{-2}{n_0}\Big)\Big)\sum_{k\mid n}\Big(\frac{-328}{k}\Big) \qquad (n = 2^{\alpha}n_0,\ 2 \nmid n_0).$$

THEOREM 9.4. *Let $n \in \mathbb{N}$. Then*

$$R([7,3,13],n) = \frac{1}{2}\Big(1-\Big(\frac{n_0}{5}\Big)\Big)\sum_{k\mid n}\Big(\frac{-355}{k}\Big) \qquad (n = 5^{\alpha}n_0,\ 5 \nmid n_0),$$

$$R([7,2,14],n) = \frac{1}{2}\Big(1-\Big(\frac{-1}{n_0}\Big)\Big)\sum_{k\mid n}\Big(\frac{-388}{k}\Big) \qquad (n = 2^{\alpha}n_0,\ 2 \nmid n_0),$$

$$R([11,2,13],n) = \frac{1}{2}\Big(1-\Big(\frac{2}{n_0}\Big)\Big)\sum_{k\mid n}\Big(\frac{-568}{k}\Big) \qquad (n = 2^{\alpha}n_0,\ 2 \nmid n_0),$$

$$R([11,9,17],n) = \frac{1}{2}\Big(1-\Big(\frac{n_0}{23}\Big)\Big)\sum_{k\mid n}\Big(\frac{-667}{k}\Big) \qquad (n = 23^{\alpha}n_0,\ 23 \nmid n_0),$$

$$R([11,5,17],n) = \frac{1}{2}\Big(1-\Big(\frac{n_0}{3}\Big)\Big)\sum_{k\mid n}\Big(\frac{-723}{k}\Big) \qquad (n = 3^{\alpha}n_0,\ 3 \nmid n_0).$$

THEOREM 9.5. *Let $n \in \mathbb{N}$. Then*

$$R([13,11,17],n) = \frac{1}{2}\Big(1-\Big(\frac{n_0}{7}\Big)\Big)\sum_{k\mid n}\Big(\frac{-763}{k}\Big) \qquad (n = 7^{\alpha}n_0,\ 7 \nmid n_0),$$

$$R([11,8,19],n) = \frac{1}{2}\Big(1-\Big(\frac{-1}{n_0}\Big)\Big)\sum_{k\mid n}\Big(\frac{-772}{k}\Big) \qquad (n = 2^{\alpha}n_0,\ 2 \nmid n_0),$$

$$R([7,5,35],n) = \frac{1}{2}\Big(1-\Big(\frac{n_0}{5}\Big)\Big)\sum_{k\mid n}\Big(\frac{-955}{k}\Big) \qquad (n = 5^{\alpha}n_0,\ 5 \nmid n_0),$$

$$R([11,3,23],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{17}\Big)\Big)\sum_{k\mid n}\Big(\frac{-1003}{k}\Big)\quad (n=17^\alpha n_0,\ 17\nmid n_0),$$

$$R([7,3,37],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{13}\Big)\Big)\sum_{k\mid n}\Big(\frac{-1027}{k}\Big)\quad (n=13^\alpha n_0,\ 13\nmid n_0).$$

Theorem 9.6. *Let $n\in\mathbb{N}$. Then*

$$R([11,7,29],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{3}\Big)\Big)\sum_{k\mid n}\Big(\frac{-1227}{k}\Big)\quad (n=3^\alpha n_0,\ 3\nmid n_0),$$

$$R([17,7,19],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{11}\Big)\Big)\sum_{k\mid n}\Big(\frac{-1243}{k}\Big)\quad (n=11^\alpha n_0,\ 11\nmid n_0),$$

$$R([13,11,29],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{19}\Big)\Big)\sum_{k\mid n}\Big(\frac{-1387}{k}\Big)\quad (n=19^\alpha n_0,\ 19\nmid n_0),$$

$$R([5,3,71],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{17}\Big)\Big)\sum_{k\mid n}\Big(\frac{-1411}{k}\Big)\quad (n=17^\alpha n_0,\ 17\nmid n_0),$$

$$R([13,1,29],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{11}\Big)\Big)\sum_{k\mid n}\Big(\frac{-1507}{k}\Big)\quad (n=11^\alpha n_0,\ 11\nmid n_0),$$

$$R([17,3,23],n)=\frac{1}{2}\Big(1-\Big(\frac{n_0}{5}\Big)\Big)\sum_{k\mid n}\Big(\frac{-1555}{k}\Big)\quad (n=5^\alpha n_0,\ 5\nmid n_0).$$

For a discriminant d and $n\in\mathbb{N}$ we recall that $\delta(n,d)=\sum_{k\mid n}(\frac{d}{k})$.

Theorem 9.7. *Let $n\in\mathbb{N}$ and $d_0\in\{-7,-19,-43,-67,-163\}$. Then*

$$R([9,3,(1-d_0)/4],n)=\begin{cases}\delta(n,d_0) & \text{if } 3\mid n-2,\\ 2\delta(n,d_0) & \text{if } 9\mid n,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let $d=9d_0$. Then we have $f(d)=3$ and $H(d)=\{[1,1,(1-d)/4],[9,9,(9-d_0)/4],[9,3,(1-d_0)/4],[9,-3,(1-d_0)/4]\}$. Clearly $H(d)\cong C_4$ and $[9,3,(1-d_0)/4]$ is a generator of $H(d)$. As $(1-d_0)/4\equiv 2 \pmod 3$, using Lemma 9.3 we see that a prime p is represented by $[9,3,(1-d_0)/4]$ if and only if $(\frac{p}{3})=-1$ and $(\frac{d_0}{p})=(\frac{p}{-d_0})=(\frac{9}{-d_0})=1$, and $p=x^2+xy+\frac{1-d}{4}y^2$ or $9x^2+9xy+\frac{9-d_0}{4}y^2$ if and only if $(\frac{p}{3})=1$ and $(\frac{d_0}{p})=0,1$. Since $h(d_0)=1$, by Lemmas 9.1, 9.4 and 9.6 we have

$$R([9,3,(1-d_0)/4],n)=\begin{cases}\frac{1}{2}\Big(1-\Big(\frac{n}{3}\Big)\Big)\sum_{k\mid n}\Big(\frac{d_0}{k}\Big) & \text{if } 3\nmid n,\\ 2\sum_{k\mid n/9}\Big(\frac{d_0}{k}\Big) & \text{if } 9\mid n,\\ 0 & \text{if } 3\parallel n.\end{cases}$$

To complete the proof, we note that if $9\mid n$ and $n=3^{\alpha}n_0$ $(3\nmid n_0)$, then

$$\sum_{k\mid n}\Big(\frac{d_0}{k}\Big)-\sum_{k\mid n/9}\Big(\frac{d_0}{k}\Big)=\sum_{k\mid n_0}\Big(\frac{d_0}{3^{\alpha-1}k}\Big)\Big(1+\Big(\frac{d_0}{3}\Big)\Big)=0. \tag{9.1}$$

THEOREM 9.8. *Let $n\in\mathbb{N}$. Then*

$$R([3,2,11],n)=\begin{cases}\delta(n,-8) & \textit{if } 4\mid n-3,\\ 2\delta(n/4,-8) & \textit{if } 16\mid n-12,\\ 2\delta(n/16,-8) & \textit{if } 16\mid n,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. It is known that $H(-128)=\{[1,0,32],[4,4,9],[3,2,11],[3,-2,11]\}$ and $f(-128)=4$. For a prime p, it is clear that $p=3x^2+2xy+11y^2$ if and only if $(\frac{-1}{p})=-1$ and $(\frac{-2}{p})=1$, and $p=x^2+32y^2$ or $4x^2+4xy+9y^2$ if and only if $(\frac{-1}{p})=(\frac{-2}{p})=1$. Thus, if $2\nmid n$, by Lemma 9.5 we have $R([3,2,11],n)=(1-(\frac{-1}{n}))\frac{2}{4}\sum_{k\mid n}(\frac{-8}{k})$. If $(n,16)=4$, then $n\equiv 4 \pmod 8$. As $H(-32)=\{[1,0,8],[3,2,3]\}$, using Lemma 9.7 we see that

$$R([3,2,11],n)=\Big(1-\Big(\frac{-1}{n/4}\Big)\Big)\sum_{k\mid n/4}\Big(\frac{-8}{k}\Big)=(1-(-1)^{(n-4)/8})\sum_{k\mid n/4}\Big(\frac{-8}{k}\Big).$$

If $(n,16)=16$, then $16\mid n$. As $h(-8)=1$, by Lemma 9.6 we have

$$R([3,2,11],n)=2\sum_{k\mid n/16}\Big(\frac{-8}{k}\Big).$$

If $(n,4^2)$ is not a square, by Lemma 9.1 we have $R([3,2,11],n)=0$. So the theorem is proved.

THEOREM 9.9. *Let $n\in\mathbb{N}$ and $n=2^{\alpha}n_0$ with $2\nmid n_0$. Let $d_1\in\{5,13,37\}$. Then*

$$R([8,4,(d_1+1)/2],n)=\begin{cases}\delta(n,-4d_1) & \textit{if } n\equiv 3 \pmod 4,\\ 2\delta(n/4,-4d_1) & \textit{if } 4\mid n \textit{ and } 4\mid n_0+1-2\alpha,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Let $d=-16d_1$. It is easily seen that $f(d)=2$ and

$$H(d)=\{[1,0,4d_1],[4,0,d_1],[8,4,(d_1+1)/2],[8,-4,(d_1+1)/2]\}.$$

Clearly $H(d)\cong C_4$ and $[8,4,(d_1+1)/2]$ is a generator of $H(d)$. For a prime p, it is clear that $p=8x^2+4xy+\frac{d_1+1}{2}y^2$ if and only if $(\frac{-1}{p})=(\frac{d_1}{p})=-1$, and

$p = x^2 + 4d_1y^2$ or $4x^2 + d_1y^2$ if and only if $(\frac{-1}{p}) = 1$ and $(\frac{d_1}{p}) = 0, 1$. Thus, if $2 \nmid n$, by Lemma 9.5 we have

$$R([8,4,(d_1+1)/2],n) = \frac{1}{2}\Big(1 - \Big(\frac{-1}{n}\Big)\Big) \sum_{k\mid n} \Big(\frac{-4d_1}{k}\Big).$$

If $2 \parallel n$, by Lemma 9.1 we have $R([8,4,(d_1+1)/2],n) = 0$. If $4 \mid n$, as $H(d/2^2) = H(-4d_1) = \{[1,0,d_1],[2,2,(d_1+1)/2]\}$, using Lemmas 2.1 and 9.1 we see that

$$R([8,4,(d_1+1)/2],n) = (1 - (-1)^{\sum_{p=2x^2+2xy+\frac{d_1+1}{2}y^2} \operatorname{ord}_p n}) \sum_{k\mid n/4} \Big(\frac{-4d_1}{k}\Big).$$

If $2 \nmid \operatorname{ord}_p(n/4)$ for some prime p with $(\frac{-4d_1}{p}) = -1$, by (2.1) we have $\sum_{k\mid n/4}(\frac{-4d_1}{k}) = 0$ and so $R([8,4,(d_1+1)/2],n) = 0$. Now assume that $2 \mid \operatorname{ord}_p(n/4)$ for every prime p with $(\frac{-4d_1}{p}) = -1$. As $p = x^2 + d_1y^2$ implies $p \equiv 1 \pmod 4$, and $p = 2x^2 + 2xy + \frac{d_1+1}{2}y^2$ implies $p = 2$ or $p \equiv 3 \pmod 4$, we see that

$$\begin{aligned}\Big(\frac{-1}{n_0}\Big) &= \prod_{p\mid n_0} \Big(\frac{-1}{p}\Big)^{\operatorname{ord}_p n_0} = \prod_{p\mid n_0} \Big(\frac{-1}{p}\Big)^{\operatorname{ord}_p(n/4)} \\ &= \prod_{p\mid n_0,\, (\frac{-4d_1}{p})=0,1} \Big(\frac{-1}{p}\Big)^{\operatorname{ord}_p(n/4)} = \prod_{p=2x^2+2xy+\frac{d_1+1}{2}y^2 \ne 2} (-1)^{\operatorname{ord}_p(n/4)} \\ &= (-1)^{\alpha} \cdot (-1)^{\sum_{p=2x^2+2xy+\frac{d_1+1}{2}y^2} \operatorname{ord}_p n}.\end{aligned}$$

So we always have

$$R([8,4,(d_1+1)/2],n) = (1 - (-1)^{\alpha + (n_0-1)/2}) \sum_{k\mid n/4} \Big(\frac{-4d_1}{k}\Big).$$

Now putting the above together we deduce the result.

Remark 9.1. As $[8,4,3] = [3,-4,8] = [3,2,7]$, we have $R([8,4,3],n) = R([3,2,7],n)$. If $4 \mid n$, $d_1 \in \{5,13\}$ and $n/4 = d_1^r n_1$ $(d_1 \nmid n_1)$, by appealing to Lemma 9.7 we have

$$R([8,4,(d_1+1)/2],n) = \Big(1 - \Big(\frac{n_1}{d_1}\Big)\Big) \sum_{k\mid n/4} \Big(\frac{-4d_1}{k}\Big).$$

Using Lemmas 2.1 and 9.1–9.7 one can similarly prove the following results.

Theorem 9.10. *Let $n \in \mathbb{N}$. Then*

$$R([5,2,10],n) = \begin{cases} \delta(n,-4) & \text{if } n \equiv 3,5,6 \pmod 7, \\ 4\delta(n/49,-4) & \text{if } 49 \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

$$R([3,1,23],n)=\begin{cases}\delta(n,-11) & \text{if } n\equiv\pm2 \pmod 5,\\ 2\delta(n/25,-11) & \text{if } 25\mid n,\\ 0 & \text{otherwise.}\end{cases}$$

THEOREM 9.11. *Let $n\in\mathbb{N}$. Then*

$$R([7,1,17],n)=\begin{cases}\delta(n,-19) & \text{if } n\equiv\pm2 \pmod 5,\\ 2\delta(n/25,-19) & \text{if } 25\mid n,\\ 0 & \text{otherwise,}\end{cases}$$

$$R([7,1,13],n)=\begin{cases}\delta(n,-3) & \text{if } n\equiv 2,6,7,8,10 \pmod{11},\\ 6\delta(n/121,-3) & \text{if } 121\mid n,\\ 0 & \text{otherwise,}\end{cases}$$

$$R([7,5,19],n)=\begin{cases}\delta(n,-3) & \text{if } n\equiv 2,5,6,7,8,11 \pmod{13},\\ 6\delta(n/169,-3) & \text{if } 169\mid n,\\ 0 & \text{otherwise.}\end{cases}$$

THEOREM 9.12. *Let $n\in\mathbb{N}$, $p\in\{3,5\}$ and $n=p^\alpha n_0$ $(p\nmid n_0)$. Then*

$$R([p+2,2,8],n)=\begin{cases}\delta(n,p(p-16)) & \text{if } 2\nmid n \text{ and } (\frac{n_0}{p})=-1,\\ \delta(n/4,p(p-16)) & \text{if } 4\mid n \text{ and } (\frac{n_0}{p})=-1,\\ 0 & \text{otherwise.}\end{cases}$$

THEOREM 9.13. *Let $n\in\mathbb{N}$. Then*

$$R([5,2,13],n)=\begin{cases}\delta(n,-4) & \text{if } n\equiv\pm3 \pmod 8,\\ 2\delta(n/4,-4) & \text{if } n\equiv\pm12 \pmod{32},\\ 2\delta(n/16,-4) & \text{if } n\equiv 16 \pmod{32},\\ 4\delta(n/64,-4) & \text{if } 64\mid n,\\ 0 & \text{otherwise.}\end{cases}$$

THEOREM 9.14. *Let $n\in\mathbb{N}$. Then*

$$R([5,4,8],n)=\begin{cases}\delta(n,-4) & \text{if } n\equiv 5 \pmod 6,\\ 2\delta(n/4,-4) & \text{if } n\equiv 8 \pmod{12},\\ 2\delta(n/9,-4) & \text{if } n\equiv 9 \pmod{18},\\ 4\delta(n/36,-4) & \text{if } 36\mid n,\\ 0 & \text{otherwise.}\end{cases}$$

THEOREM 9.15. *Let $n\in\mathbb{N}$. Then*

$$R([8,6,9],n)=\begin{cases}\delta(n,-7) & \text{if } n\equiv 5 \pmod 6,\\ \delta(n/4,-7) & \text{if } n\equiv 8 \pmod{12},\\ 2\delta(n/9,-7) & \text{if } n\equiv 9 \pmod{18},\\ 2\delta(n/36,-7) & \text{if } 36\mid n,\\ 0 & \text{otherwise.}\end{cases}$$

THEOREM 9.16. *Let $n \in \mathbb{N}$. Then*

$$R([8,4,13],n) = \begin{cases} \delta(n,-4) & \text{if } n \equiv \pm 3 \pmod{10}, \\ 2\delta(n/4,-4) & \text{if } n \equiv \pm 8 \pmod{20}, \\ 2\delta(n/25,-4) & \text{if } n \equiv 25 \pmod{50}, \\ 4\delta(n/100,-4) & \text{if } 100 \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

10. Formulas for $t_n(1,8)$, $t_n(1,63)$, $t_n(7,9)$, $t_n(1,55)$, $t_n(5,11)$, $t_n(1,39)$ and $t_n(3,13)$. For $k = 1,\dots,12$ let

$$q\prod_{m=1}^{\infty}\{(1-q^{km})(1-q^{(24-k)m})\} = \sum_{n=1}^{\infty}\phi_k(n)q^n \qquad (|q|<1).$$

In [7], for $k = 1,2,3,4,6,8,12$ we showed that $\phi_k(n)$ is a multiplicative function of n and determined the value of $\phi_k(n)$. See [7, Theorems 4.4 and 4.5].

THEOREM 10.1. *Suppose $n \in \mathbb{N}$. Then*

$$4t_n(1,8) = \sum_{k\mid 8n+9}\left(\frac{-2}{k}\right) - \phi_8(8n+9).$$

Proof. From Theorem 1.1 we know that $4t_n(1,8) = R([4,4,9],8n+9)$. As $H(-128) = \{[1,0,32],[4,4,9],[3,2,11],[3,-2,11]\} \cong C_4$ and $f(-128) = 4$, we have $R([1,0,32],8n+9) + R([4,4,9],8n+9) = N(8n+9,-128) - 2R([3,2,11],8n+9)$. On the other hand, by [7, Theorem 2.2] we have $R([1,0,32],8n+9) - R([4,4,9],8n+9) = 2\phi_8(8n+9)$. Thus

$$\begin{aligned} 4t_n(1,8) &= R([4,4,9],8n+9) \\ &= \tfrac{1}{2}N(8n+9,-128) - R([3,2,11],8n+9) - \phi_8(8n+9). \end{aligned}$$

By Lemma 2.1 we have $N(8n+9,-128) = 2\sum_{k\mid 8n+9}(\frac{-8}{k})$. By Theorem 9.8 we have $R([3,2,11],8n+9) = 0$. Thus the result follows.

THEOREM 10.2. *Suppose $n \in \mathbb{N}$.*

(i) *If $n+8 = 2^{\alpha_0}n_0$ with $2 \nmid n_0$, then*

$$t_n(1,63) = \begin{cases} \displaystyle\sum_{k\mid n_0}\left(\frac{k}{7}\right) & \text{if } 9 \mid n-1, \\ \displaystyle\frac{1}{2}\sum_{k\mid n_0}\left(\frac{k}{7}\right) & \text{if } 3 \mid n \text{ and } 2 \mid \alpha_0, \\ \displaystyle\frac{1}{2}\Big(\sum_{k\mid n_0}\left(\frac{k}{7}\right) + (-1)^{(\alpha_0+1)/2}\phi_3(n_0)\Big) & \text{if } 6 \mid n \text{ and } 2 \nmid \alpha_0, \\ 0 & \text{if } 3 \nmid n \text{ and } 9 \nmid n-1. \end{cases}$$

(ii) *If* $n+2=2^{\alpha_1}n_1$ *with* $2\nmid n_1$, *then*

$$t_n(7,9)=\begin{cases}\sum_{k\mid n_1}\left(\frac{k}{7}\right) & \text{if } 9\mid n+2,\\ \frac12\sum_{k\mid n_1}\left(\frac{k}{7}\right) & \text{if } 3\mid n \text{ and } 2\mid\alpha_1,\\ \frac12\Big(\sum_{k\mid n_1}\left(\frac{k}{7}\right)+(-1)^{(\alpha_1-1)/2}\phi_3(n_1)\Big) & \text{if } 6\mid n \text{ and } 2\nmid\alpha_1,\\ 0 & \text{if } 3\nmid n \text{ and } 9\nmid n+2.\end{cases}$$

Proof. From Theorem 1.1 we see that

$$4t_n(1,63)=\begin{cases}R([1,1,16],2n+16) & \text{if } 2\nmid n,\\ R([1,1,16],2n+16)-R([1,1,16],n/2+4) & \text{if } 2\mid n,\end{cases}$$
$$4t_n(7,9)=\begin{cases}R([7,7,4],2n+4) & \text{if } 2\nmid n,\\ R([7,7,4],2n+4)-R([7,7,4],n/2+1) & \text{if } 2\mid n.\end{cases}$$

Observe that $R([7,7,4],m)=R([4,-7,7],m)=R([4,1,4],m)$. We then have

$$4t_n(7,9)=\begin{cases}R([4,1,4],2n+4) & \text{if } 2\nmid n,\\ R([4,1,4],2n+4)-R([4,1,4],n/2+1) & \text{if } 2\mid n.\end{cases}$$

As $H(-63)=\{[1,1,16],[4,1,4],[2,1,8],[2,-1,8]\}\cong C_4$, we have

$$R([1,1,16],m)+R([4,1,4],m)=N(m,-63)-2R([2,1,8],m).$$

On the other hand, by [7, Theorem 2.2], $R([1,1,16],m)-R([4,1,4],m)=2\phi_3(m)$. Thus,

$$R([1,1,16],m)=\tfrac12N(m,-63)-R([2,1,8],m)+\phi_3(m),$$
$$R([4,1,4],m)=\tfrac12N(m,-63)-R([2,1,8],m)-\phi_3(m).$$

From Lemma 2.1 and (9.1) we see that

$$N(m,-63)=\begin{cases}2\sum_{k\mid m}\left(\frac{-7}{k}\right) & \text{if } 3\nmid m,\\ 8\sum_{k\mid m/9}\left(\frac{-7}{k}\right)=8\sum_{k\mid m}\left(\frac{-7}{k}\right) & \text{if } 9\mid m,\\ 0 & \text{otherwise.}\end{cases}$$

By Theorem 9.7 we have

$$\begin{aligned}R([2,1,8],m)&=R([2,-3,9],m)=R([9,3,2],m)\\&=\begin{cases}\sum_{k\mid m}\left(\frac{-7}{k}\right)&\text{if } 3\mid m-2,\\ 2\sum_{k\mid m}\left(\frac{-7}{k}\right)&\text{if } 9\mid m,\\ 0&\text{otherwise.}\end{cases}\end{aligned}$$

Thus, for $m=2^tm_0$ $(2\nmid m_0)$ we have

$$\begin{aligned}&\tfrac12N(m,-63)-R([2,1,8],m)\\&=\begin{cases}\sum_{k\mid m}\left(\frac{-7}{k}\right)=\sum_{k\mid m_0}\sum_{i=0}^{t}\left(\frac{-7}{2^ik}\right)=(t+1)\sum_{k\mid m_0}\left(\frac{k}{7}\right)&\text{if } 3\mid m-1,\\ 2\sum_{k\mid m}\left(\frac{-7}{k}\right)=2\sum_{k\mid m_0}\sum_{i=0}^{t}\left(\frac{-7}{2^ik}\right)=2(t+1)\sum_{k\mid m_0}\left(\frac{k}{7}\right)&\text{if } 9\mid m,\\ 0&\text{otherwise.}\end{cases}\end{aligned}$$

Hence, for $\alpha,m_0\in\mathbb{N}$ with $2\nmid m_0$ we have

$$\begin{aligned}&\tfrac12N(2^{\alpha+1}m_0,-63)-R([2,1,8],2^{\alpha+1}m_0)\\&\quad-\tfrac12N(2^{\alpha-1}m_0,-63)+R([2,1,8],2^{\alpha-1}m_0)\\&\quad=\begin{cases}(\alpha+2-\alpha)\sum_{k\mid m_0}\left(\frac{k}{7}\right)=2\sum_{k\mid m_0}\left(\frac{k}{7}\right)&\text{if } 3\mid 2^\alpha m_0+1,\\ (2(\alpha+2)-2\alpha)\sum_{k\mid m_0}\left(\frac{k}{7}\right)=4\sum_{k\mid m_0}\left(\frac{k}{7}\right)&\text{if } 9\mid m_0,\\ 0&\text{otherwise.}\end{cases}\end{aligned}$$

If $2\nmid n$, by [7, Theorem 4.5(i)] we have $\phi_3(2n+16)=\phi_3(2n+4)=0$. Hence, applying the above we obtain

$$\begin{aligned}4t_n(1,63)&=R([1,1,16],2n+16)\\&=\tfrac12N(2n+16,-63)-R([2,1,8],2n+16)\\&=\begin{cases}2\sum_{k\mid n+8}\left(\frac{k}{7}\right)&\text{if } 3\mid n,\\ 4\sum_{k\mid n+8}\left(\frac{k}{7}\right)&\text{if } 9\mid n-1,\\ 0&\text{otherwise}\end{cases}\end{aligned}$$

and

$$\begin{aligned}4t_n(7,9) &= R([4,1,4],2n+4)\\ &= \tfrac{1}{2}N(2n+4,-63) - R([2,1,8],2n+4)\\ &= \begin{cases} 2\sum_{k\mid n+2}\left(\frac{k}{7}\right) & \text{if } 3\mid n,\\ 4\sum_{k\mid n+2}\left(\frac{k}{7}\right) & \text{if } 9\mid n+2,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now assume $2\mid n$. Suppose $n = 2^{\alpha_0}n_0 - 8 = 2^{\alpha_1}n_1 - 2$ with $2\nmid n_0n_1$. From the above we deduce

$$\begin{aligned}4t_n(1,63) &= R([1,1,16],2^{\alpha_0+1}n_0) - R([1,1,16],2^{\alpha_0-1}n_0)\\ &= \tfrac{1}{2}N(2^{\alpha_0+1}n_0,-63) - R([2,1,8],2^{\alpha_0+1}n_0) - \tfrac{1}{2}N(2^{\alpha_0-1}n_0,-63)\\ &\quad + R([2,1,8],2^{\alpha_0-1}n_0) + \phi_3(2^{\alpha_0+1}n_0) - \phi_3(2^{\alpha_0-1}n_0)\\ &= \begin{cases} 2\sum_{k\mid n_0}\left(\frac{k}{7}\right) + \phi_3(2^{\alpha_0+1}n_0) - \phi_3(2^{\alpha_0-1}n_0) & \text{if } 3\mid n,\\ 4\sum_{k\mid n_0}\left(\frac{k}{7}\right) + \phi_3(2^{\alpha_0+1}n_0) - \phi_3(2^{\alpha_0-1}n_0) & \text{if } 9\mid n-1,\\ \phi_3(2^{\alpha_0+1}n_0) - \phi_3(2^{\alpha_0-1}n_0) & \text{otherwise}\end{cases}\end{aligned}$$

and

$$\begin{aligned}4t_n(7,9) &= R([4,1,4],2^{\alpha_1+1}n_1) - R([4,1,4],2^{\alpha_1-1}n_1)\\ &= \tfrac{1}{2}N(2^{\alpha_1+1}n_1,-63) - R([2,1,8],2^{\alpha_1+1}n_1) - \tfrac{1}{2}N(2^{\alpha_1-1}n_1,-63)\\ &\quad + R([2,1,8],2^{\alpha_1-1}n_1) - \phi_3(2^{\alpha_1+1}n_1) + \phi_3(2^{\alpha_1-1}n_1)\\ &= \begin{cases} 2\sum_{k\mid n_1}\left(\frac{k}{7}\right) - \phi_3(2^{\alpha_1+1}n_1) + \phi_3(2^{\alpha_1-1}n_1) & \text{if } 3\mid n,\\ 4\sum_{k\mid n_1}\left(\frac{k}{7}\right) - \phi_3(2^{\alpha_1+1}n_1) + \phi_3(2^{\alpha_1-1}n_1) & \text{if } 9\mid n+2,\\ -\phi_3(2^{\alpha_1+1}n_1) + \phi_3(2^{\alpha_1-1}n_1) & \text{otherwise.}\end{cases}\end{aligned}$$

As $H(-63) = \{[1,1,16],[4,1,4],[2,1,8],[2,-1,8]\} \cong C_4$ and $\phi_3(m) = \frac{1}{2}(R([1,1,16],m) - R([4,1,4],m))$, using [6, Theorem 7.4(ii)] we see that $\phi_3(m)$ is a multiplicative function of m. By [6, Theorem 8.7], for $t\in\mathbb{N}$ we have

$$\phi_3(2^t) = \begin{cases}(-1)^{t/2} & \text{if } 2\mid t,\\ 0 & \text{if } 2\nmid t.\end{cases}$$

Thus, for $i = 0, 1$,

$$\begin{aligned}\phi_3(2^{\alpha_i+1}n_i) - \phi_3(2^{\alpha_i-1}n_i) &= (\phi_3(2^{\alpha_i+1}) - \phi_3(2^{\alpha_i-1}))\phi_3(n_i)\\ &= \begin{cases} 2(-1)^{(\alpha_i+1)/2}\phi_3(n_i) & \text{if } 2\nmid \alpha_i,\\ 0 & \text{if } 2 \mid \alpha_i.\end{cases}\end{aligned}$$

From [7, Theorem 4.5(i)] we know that $\phi_3(n_i) = 0$ for $n_i \equiv 0, 2 \pmod 3$. As $n \equiv 1 \pmod 3$ implies $3 \mid n_i$ and so $\phi_3(n_i) = 0$, and $n \equiv 2 \pmod 3$ and $2 \nmid \alpha_i$ implies $n_i \equiv 2 \pmod 3$ and so $\phi_3(n_i) = 0$, we see that $\phi_3(2^{\alpha_i+1}n_i) - \phi_3(2^{\alpha_i-1}n_i) = 0$ for $n \not\equiv 0 \pmod 3$. Now putting all the above together we deduce the result.

Theorem 10.3. *Suppose $n \in \mathbb{N}$. Then $\phi_3(n) = t_{2n-2}(7, 9) - t_{2n-8}(1, 63)$ and so $t_{2n-2}(7, 9) - t_{2n-8}(1, 63)$ is a multiplicative function of n.*

Proof. Suppose $2n = 2^\alpha n_0$ with $2 \nmid n_0$. According to the proof of Theorem 10.2, $\phi_3(n)$ is a multiplicative function of n and

$$\phi_3(n) = \phi_3(2^{\alpha-1}n_0) = \phi_3(2^{\alpha-1})\phi_3(n_0) = \begin{cases} (-1)^{(\alpha-1)/2}\phi_3(n_0) & \text{if } 2 \nmid \alpha,\\ 0 & \text{if } 2 \mid \alpha.\end{cases}$$

As $\phi_3(1) = 1$, $\phi_3(2) = \phi_3(3) = 0$ and $\phi_3(4) = -1$, we see that $\phi_3(n) = t_{2n-2}(7, 9) - t_{2n-8}(1, 63)$ for $n = 1, 2, 3, 4$. Now suppose $n > 4$. From the above and Theorem 10.2 we deduce

$$\begin{aligned}&t_{2n-2}(7, 9) - t_{2n-8}(1, 63)\\ &\qquad = \begin{cases} (-1)^{(\alpha-1)/2}\phi_3(n_0) = \phi_3(n) & \text{if } 3 \mid n-1 \text{ and } 2 \nmid \alpha,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

If $3 \mid n-1$ and $2 \mid \alpha$, then $\alpha \ge 2$ and so $\phi_3(n) = 0$ by the above. From [7, Theorem 4.5(i)] we also have $\phi_3(n) = 0$ for $n \equiv 0, 2 \pmod 3$. Thus we always have $\phi_3(n) = t_{2n-2}(7, 9) - t_{2n-8}(1, 63)$. So the theorem is proved.

Theorem 10.4. *Suppose $n \in \mathbb{N}$, $m \in \{3, 5\}$ and $n + m + 2 = 4^r m^s A$ with $(A, 2m) = 1$. Then*

$$t_n(1, m(16-m)) = t_{n+m}(m, 16-m) = \frac{1 - (\frac{A}{m})}{4}\sum_{k \mid A}\Big(\frac{-m(16-m)}{k}\Big).$$

Proof. From Theorem 1.1 we see that

$$\begin{aligned}&4t_n(1, 8m + 15)\\ &\quad = R([1, 1, 2m + 4], 2n + 2m + 4) - R([1, 0, 8m + 15], 2n + 2m + 4)\end{aligned}$$

and

$$4t_n(m, 16-m) = R([m, m, 4], 2n + 4) - R([m, 0, 16-m], 2n + 4).$$

As $[1,0,8m+15]=[1,2,4(2m+4)]$, $[m,0,16-m]=[m,2m,16]$ and $f(-4(8m+15))=2$, by [6, Theorem 3.2] we have

$$R([1,0,8m+15],2n+2m+4)=\begin{cases}0 & \text{if } 2\mid n,\\ R([1,1,2m+4],(n+m+2)/2) & \text{if } 2\nmid n\end{cases}$$

and

$$R([m,0,16-m],2n+4)=\begin{cases}R([m,m,4],(n+2)/2) & \text{if } 2\mid n,\\ 0 & \text{if } 2\nmid n.\end{cases}$$

Hence

$$\begin{aligned}(10.1)\qquad & 4t_n(1,8m+15)\\ &=\begin{cases}R([1,1,2m+4],2n+2m+4) & \text{if } 2\mid n,\\ R([1,1,2m+4],2n+2m+4) & \\ \quad -R([1,1,2m+4],(n+m+2)/2) & \text{if } 2\nmid n\end{cases}\end{aligned}$$

and

$$\begin{aligned}(10.2)\qquad & 4t_{n+m}(m,16-m)\\ &=\begin{cases}R([m,m,4],2n+2m+4) & \text{if } 2\mid n,\\ R([m,m,4],2n+2m+4)-R([m,m,4],(n+m+2)/2) & \text{if } 2\nmid n.\end{cases}\end{aligned}$$

It is easily seen that

$$H(-8m-15)=\{[1,1,2m+4],[m,m,4],[2,1,m+2],[2,-1,m+2]\}\cong C_4.$$

Thus applying [6, Theorem 7.4(ii)] we see that $F(n)=\frac{1}{2}(R([1,1,2m+4],n)-R([m,m,4],n))$ is multiplicative. Hence

$$F(2n+2m+4)=F(2^{2r+1}n_0)=F(2^{2r+1})F(n_0)$$

and

$$F((n+m+2)/2)=F(2^{2r-1}n_0)=F(2^{2r-1})F(n_0)\quad \text{for}\quad r\ge 1.$$

Since $f(-8m-15)=1$ and 2 is represented by $[2,1,m+2]$, using [6, Theorem 8.7] we see that $F(2^t)=0$ for $2\nmid t$. Thus $F((n+m+2)/2)=0$ for $r\ge 1$ and $F(2n+2m+4)=0$. Hence $R([1,1,2m+4],(n+m+2)/2)$ $=R([m,m,4],(n+m+2)/2)$ for $r\ge 1$ and $R([1,1,2m+4],2n+2m+4)$ $=R([m,m,4],2n+2m+4)$. This together with (10.1) and (10.2) yields $t_n(1,8m+15)=t_{n+m}(m,16-m)$. From the above, Lemma 2.1 and Theorem 9.1 we deduce

$$\begin{aligned}
&2R([1,1,2m+4],2n+2m+4)\\
&\quad= R([1,1,2m+4],2n+2m+4)+R([m,m,4],2n+2m+4)\\
&\quad= N(2n+2m+4,-8m-15)-2R([2,1,m+2],2n+2m+4)\\
&\quad= 2\sum_{k\mid 2n+2m+4}\Big(\frac{-8m-15}{k}\Big)-\Big(1-\Big(\frac{2^{2r+1}A}{m}\Big)\Big)\sum_{k\mid 2n+2m+4}\Big(\frac{-8m-15}{k}\Big)\\
&\quad= \Big(1-\Big(\frac{A}{m}\Big)\Big)\sum_{k\mid 2n+2m+4}\Big(\frac{-8m-15}{k}\Big).
\end{aligned}$$

Similarly, for odd n we have

$$2R\Big([1,1,2m+4],\frac{n+m+2}{2}\Big)=\Big(1-\Big(\frac{A}{m}\Big)\Big)\sum_{k\mid (n+m+2)/2}\Big(\frac{-8m-15}{k}\Big).$$

If $2\mid n$, from the above, (10.1) and the fact that $8m+15=m(16-m)$ we obtain

$$\begin{aligned}
t_n(1,m(16-m))&=t_n(1,8m+15)\\
&=\tfrac14 R([1,1,2m+4],2n+2m+4)\\
&=\frac{1-(\frac{A}{m})}{8}\sum_{k\mid 2n+2m+4}\Big(\frac{-m(16-m)}{k}\Big)\\
&=\frac{1-(\frac{A}{m})}{8}\sum_{k\mid m^sA}\Big(\Big(\frac{-m(16-m)}{k}\Big)+\Big(\frac{-m(16-m)}{2k}\Big)\Big)\\
&=\frac{1-(\frac{A}{m})}{4}\sum_{k\mid m^sA}\Big(\frac{-m(16-m)}{k}\Big)=\frac{1-(\frac{A}{m})}{4}\sum_{k\mid A}\Big(\frac{-m(16-m)}{k}\Big).
\end{aligned}$$

If $2\nmid n$, by (10.1) and the above we have

$$\begin{aligned}
&t_n(1,m(16-m))=t_n(1,8m+15)\\
&\quad=\tfrac14\big(R([1,1,2m+4],2n+2m+4)-R([1,1,2m+4],(n+m+2)/2)\big)\\
&\quad=\frac{1-(\frac{A}{m})}{8}\Big(\sum_{k\mid 2n+2m+4}\Big(\frac{-m(16-m)}{k}\Big)-\sum_{k\mid (n+m+2)/2}\Big(\frac{-m(16-m)}{k}\Big)\Big)\\
&\quad=\frac{1-(\frac{A}{m})}{8}\sum_{k\mid m^sA}\Big(\Big(\frac{-m(16-m)}{2^{2r}k}\Big)+\Big(\frac{-m(16-m)}{2^{2r+1}k}\Big)\Big)\\
&\quad=\frac{1-(\frac{A}{m})}{4}\sum_{k\mid m^sA}\Big(\frac{-m(16-m)}{k}\Big)=\frac{1-(\frac{A}{m})}{4}\sum_{k\mid A}\Big(\frac{-m(16-m)}{k}\Big).
\end{aligned}$$

This completes the proof.

THEOREM 10.5. *Let $n \in \mathbb{N}$, $m \in \{3,5\}$ and $f(n) = t_{2n-2}(m, 16-m) - t_{2n-2-m}(1, m(16-m))$. Then $f(n)$ is a multiplicative function of n.*

Proof. Define $F(n) = \frac{1}{2}(R([1,1,2m+4],n) - R([m,m,4],n))$. Since $H(-8m-15) = \{[1,1,2m+4],[m,m,4],[2,1,m+2],[2,-1,m+2]\} \cong C_4$, from [6, Theorem 7.4(ii)] we know that $F(n)$ is multiplicative. It is easily seen that $F(1) = 1$, $F(2) = F(3) = 0$ and so $f(n) = F(n)$ for $n = 1,2,3$. From [6, Theorem 8.7] we see that $F(2^t) = (-1)^{t/2}$ or 0 according as $2 \mid t$ or $2 \nmid t$. Suppose $n = 2^\alpha n_0$ with $2 \nmid n_0$. We then have $F(2^{\alpha+2}) = -F(2^\alpha)$. For $n > 3$, from (10.1), (10.2) and the above we derive

$$\begin{aligned}4t_{2n-2-m}(1, m(16-m)) - 4t_{2n-2}(m, 16-m) &= 2F(4n) - 2F(n)\\ = 2(F(2^{\alpha+2}) - F(2^\alpha))F(n_0) &= -4F(2^\alpha)F(n_0) = -4F(n).\end{aligned}$$

Thus, $f(n) = F(n)$. This proves the theorem.

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Zhi-Hong Sun
School of Mathematical Sciences
Huaiyin Normal University
Huaian, Jiangsu 223001, P.R. China
E-mail: szh6174@yahoo.com
http://www.hytc.edu.cn/xsjl/szh

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